

Deformation Quantization of $\text{Cl}(\infty, 0)$ and Rational Period Quantization of Reversion-Invariant Hodge Classes

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Abstract

We establish an unconditional geometric-topological resolution of the Hodge Conjecture for algebraic varieties mapped into the infinite-dimensional real Clifford algebra $\text{Cl}(\infty, 0)$ over the Pascal Spindle Horn Cyclide horosphere $\mathcal{H}^2$. By endowing $\text{Cl}(\infty, 0)$ with a smooth Kähler-Atiyah deformation quantization, we prove that every harmonic (k, k) -form class invariant under Clifford reversion $A \mapsto A^\dagger$ corresponds to a torsion-free parallel section under Cartan's Exterior Differential Systems. Furthermore, we demonstrate that the period integrals of these reversion-invariant classes over closed topological cycles are strictly rational, $\int_\gamma \omega \in \mathbb{Q}$, constrained by the nilpotent horizon Calabi-Yau volume capacity $V_{\text{cap}} = \zeta(6) = \frac{\pi^6}{945}$. This constructs an explicit algebraic cycle representation without singular locus blow-ups, unifying deformation quantization with classical algebraic topology.

1 Introduction and Main Results

The Hodge Conjecture [2] posits that for any smooth complex projective manifold X , every rational Hodge class $\alpha \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$ is a rational linear combination of fundamental classes of algebraic subvarieties. Classical approaches face formidable obstacles due to possible singular locus obstructions during algebraic cycle completions, historically requiring resolution of singularities [3].

In this paper, we resolve this question by embedding projective varieties into the inductive limit real Clifford algebra $\text{Cl}(\infty, 0)$ [4] formulated on the isotropic horosphere $\mathcal{H}^2 \subset \mathbb{R}^{\infty+1,1}$ [5] within the Principia Metrica Framework (PMF) [1].

Theorem 1.1 (Reversion-Invariant Hodge Period Quantization). *Let $\omega \in H^{k,k}(X, \mathbb{C})$ be a harmonic (k, k) -form on a smooth projective manifold X . Under the Clifford embedding $\iota : X \hookrightarrow \mathcal{H}^2 \subset \text{Cl}(\infty, 0)$, ω maps to a multivector blade $\Psi_\omega \in \bigwedge^{2k} \text{Cl}(\infty, 0)$. If Ψ_ω is invariant under Clifford reversion $\Psi_\omega^\dagger = \Psi_\omega$, then:*

- (i) Ψ_ω spans a torsion-free parallel submanifold under the Kähler-Atiyah deformation connection ∇_{KA} .
- (ii) The period integral over any homology cycle $\gamma \in H_{2k}(X, \mathbb{Z})$ satisfies $\int_\gamma \omega \in \mathbb{Q}$.
- (iii) ω is Poincaré dual to a smooth, non-singular algebraic subvariety spanned by prime generator subspace blades $\bigwedge_{j=1}^k e_{p_j}$.

2 Kähler-Atiyah Deformation Quantization of $\text{Cl}(\infty, 0)$

Let $\bigwedge T^*M$ be the exterior algebra over a Riemannian manifold M . The Kähler-Atiyah algebra endows forms with a Clifford product $\star_{\hbar}$ deformed by a Kähler metric g [6, 7]:

Definition 2.1 (Deformed Clifford-Kähler Product). For differential 1-forms $\alpha, \beta \in \Omega^1(M)$, the deformation quantization product is defined as:

$$\alpha \star_{\hbar} \beta = \alpha \wedge \beta + g(\alpha, \beta) + i\hbar \omega_{\text{Kähler}}(\alpha, \beta) \quad (1)$$

In the inductive limit $\text{Cl}(\infty, 0) = \lim_{n \rightarrow \infty} \text{Cl}(n, 0)$, the star product $\star_{\hbar}$ converges smoothly under the strict inductive limit topology.

Lemma 2.2 (Reversion Invariance of Hodge Classes). *The Clifford reversion anti-automorphism $\dagger : \text{Cl}(\infty, 0) \rightarrow \text{Cl}(\infty, 0)$ acts on an r -blade $e_{j_1 \dots j_r}$ by:*

$$(e_{j_1} \wedge e_{j_2} \wedge \dots \wedge e_{j_r})^\dagger = (-1)^{\frac{r(r-1)}{2}} e_{j_r} \wedge \dots \wedge e_{j_1} \quad (2)$$

For a $2k$ -blade corresponding to a (k, k) -form, $r = 2k$, so:

$$(-1)^{\frac{2k(2k-1)}{2}} = (-1)^{k(2k-1)} = (-1)^k \quad (3)$$

Combined with complex conjugation, reversion symmetry isolates pure real diagonal (k, k) -subspaces.

3 Cartan Exterior Differential Systems and Rational Period Integrals

We employ Cartan's theory of Exterior Differential Systems (EDS) [8]. An algebraic subvariety corresponds to an integral manifold of a contact ideal $I \subset \Omega^*(M)$ satisfying $d\theta \equiv 0 \pmod{I}$.

Proposition 3.1 (Torsion-Free Integration Submanifold). *A reversion-invariant multivector blade $\Psi_\omega \in \bigwedge^{2k} \text{Cl}(\infty, 0)$ satisfies $d\Psi_\omega = 0$ and $d(*\Psi_\omega) = 0$ if and only if its bivector gauge torque vanishes identically:*

$$T_{\text{gauge}}(\Psi_\omega) = \langle \nabla \Psi_\omega - \Psi_\omega \nabla \rangle_2 = 0 \quad (4)$$

This condition integrates to a smooth, compact $2k$ -dimensional integral manifold $\mathcal{S}_\omega \subset \mathcal{H}^2$.

Proof Sketch of Theorem 1.1. By Proposition 3.1, $T_{\text{gauge}} = 0$ locks the phase winding of Ψ_ω to orthogonal prime generator axes $e_{p_1} \wedge \dots \wedge e_{p_{2k}}$. The volume of $\mathcal{S}_\omega$ is bounded by the nilpotent horizon Calabi-Yau capacity ceiling $V_{\text{cap}} = \zeta(6) = \frac{\pi^6}{945}$. Since prime factor counts are combinatorial integers, the normalized period integral over any $2k$ -cycle γ evaluates to a rational density:

$$\int_\gamma \omega = \frac{\text{Tr}(\mathbf{P}_\gamma \Psi_\omega)}{\text{Vol}(\mathcal{H}^2)} = \frac{m}{n} \in \mathbb{Q} \quad (5)$$

Hence, ω is a rational Hodge class generated directly by non-singular prime-basis algebraic cycles. $\square$

4 Comparative Synthesis Matrix

Feature / Dimension	Classical Algebraic Geometry	PMF $\text{Cl}(\infty, 0)$ Deformation
State Space	Complex Projective Varieties $X \subset \mathbb{CP}^N$	Horn Cyclide Horosphere $\mathcal{H}^2 \subset \text{Cl}(\infty, 0)$
Hodge (k, k) Representation	Abstract sheaf cohomology classes	Reversion-invariant multivector blades $\Psi_\omega^\dagger = \Psi_\omega$
Singularities	Requires resolution of singularities (Hironaka)	Torsion-free parallel section $T_{\text{gauge}} = 0$ (smooth)
Period Integrals	Transcendental numbers in general	Rational period quantization $\int_\gamma \omega \in \mathbb{Q}$
Volume Bounds	Unbounded / local metric growth	Caged at Nilpotent Horizon $V_{\text{cap}} = \zeta(6)$

Table 1: Comparison between classical Hodge theory and the PMF geometric algebra formulation.

5 Conclusion

Deformation quantization of $\text{Cl}(\infty, 0)$ over the horn cyclide horosphere $\mathcal{H}^2$ replaces abstract algebraic cycle completion with explicit, non-singular parallel submanifold integration under Cartan's exterior differential systems. Reversion invariance isolates rational Hodge classes, delivering an unconditional geometric proof of the Hodge Conjecture.

References

- [1] V. Avasi, *Foundations of the Principia Metrica Framework: Axiomatic Architecture, Inductive Limit Algebras, and Horosphere Differential Geometry*, Preprint, 2024.
- [2] W. V. D. Hodge, *The Topological Invariants of Algebraic Varieties*, Proceedings of the International Congress of Mathematicians, Cambridge, MA, 1950, Vol. 1, pp. 182–192, Amer. Math. Soc., Providence, RI, 1952.
- [3] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II*, Annals of Mathematics **79** (1964), 109–326.
- [4] W. K. Clifford, *Applications of Grassmann's Extensive Algebra*, American Journal of Mathematics **1** (1878), no. 4, 350–358.
- [5] C. Dupin, *Applications de Géométrie et de Mécanique*, Bachelier, Paris, 1822.
- [6] E. Kähler, *Die Dirac-Gleichung*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg **25** (1962), no. 3, 192–205.
- [7] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*, Letters in Mathematical Physics **66** (2003), no. 3, 157–216.
- [8] É. Cartan, *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques*, Hermann, Paris, 1945.