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# ON THE GENERAL ERDŐS-MOSER EQUATION VIA THE NOTION OF OLLOIDS

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ABSTRACT. We introduce and develop the notion of the **olloid**. We apply this notion to study a variant and a generalized version of the Erdős-Moser equation under some special local condition.

## 1. Introduction

The Erdős-Moser equation is an equation of the form

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

where  $m$  and  $k$  are positive integers. The only known solution to the equation is  $1^1 + 2^1 = 3^1$  and Paul Erdős is known to have conjectured that the equation has no further solution. The exponent  $k$  and the arguments in the the Erdős-Moser equation has also been studied quite extensively. In other words, several constraints on the exponent  $k$  and the argument  $m$  of the Erdős-Moser equation have been studied under a presumption that other solutions - if any - exists. In particular, it has been shown that  $k$  must be divisible by 2 and that there is no solution with  $m < 10^{1000000}$  [1]. The methods introduced by Moser were later refined and adapted to show that  $m > 1.485 \times 10^{9321155}$  [2]. This was improved to the lower bound  $m > 2.7139 \times 10^{1,667,658,416}$  in [5] via large scale computation of  $\ln(2)$ . It is also shown (see [3]) that  $6 \leq k + 2 < m < 2k$ . It is also known that  $\text{lcm}(1, 2, \dots, 200)$  must divide  $k$  and that any prime factor of  $m + 1$  must be irregular and  $> 1000$  [4]. In 2002, it was shown that all primes  $200 < p < 1000$  must divide the exponent  $k$  in the Erdős-Moser equation

$$1^k + 2^k + \cdots + m^k = (m+1)^k$$

where  $m$  and  $k$  are positive integers.

In this paper we introduce and study the notion of the **olloid** and develop a technique for extending the solution of the generalized Erdős-Moser equation upto exponents  $k$  under some special local conditions of the underlying generator. In particular, we obtain the following result

**Theorem 1.1** (The generalized extension method). *Let  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  have continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$ . If the equation*

$$\sum_{i=1}^s h(i)^k = h(s+1)^k$$

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for  $k > 1$  has a solution and there exist some  $r \in \mathbb{N}$  such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{h(i)}{h(s+1)}$$

for  $1 \leq i \leq s$ . Then the equation

$$\sum_{i=1}^s h(i)^{k+r} = h(s+1)^{k+r}$$

also has a solution.

This result is a consequence of the more fundamental result using the notion of the **olloid**.

**Lemma 1.2** (Expansion principle). *Let  $\mathbb{F}_s^k$  be an  $s$ -dimensional **olloid** of degree  $k$  for a fixed  $k \in \mathbb{N}$  with  $k > 1$ . If  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is a generator with continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$  such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for  $r \in \mathbb{N}$  then  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is also a generator of the **olloid**  $\mathbb{F}_s^{k+r}$  of degree  $k+r$ .

## 2. The notion of the olloid

In this section we launch the notion of the **olloid** and prove a fundamental lemma, which will be relevant for our studies in the sequel.

**Definition 2.1.** Let  $\mathbb{F}_s^k := \left\{ (u_1, u_2, \dots, u_s) \in \mathbb{R}^s \mid \sum_{i=1}^s u_i^k = 1, k > 1 \right\}$ . Then we call  $\mathbb{F}_s^k$  an  $s$ -dimensional **olloid** of degree  $k > 1$ . We say  $g : \mathbb{N} \rightarrow \mathbb{R}$  is a generator of the  $s$ -dimensional olloid of degree  $k$  if there exists some vector  $(v_1, v_2, \dots, v_s) \in \mathbb{F}_s^k$  such that  $v_i = g(i)$  for each  $1 \leq i \leq s$ .

*Question 2.2.* Does there exist a fixed generator  $g : \mathbb{N} \rightarrow \mathbb{R}$  with infinitely many olloids?

*Remark 2.3.* While it may be difficult to provide a general answer to question 2.2, we can in fact provide an answer by imposing certain conditions for which the generator of the **olloid** must satisfy. In particular, we launch a basic and a fundamental principle relevant for our studies in the sequel.

**Lemma 2.4** (Expansion principle). *Let  $\mathbb{F}_s^k$  be an  $s$ -dimensional **olloid** of degree  $k > 1$  for a fixed  $k \in \mathbb{N}$ . If  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is a generator with continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$  such that*

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for  $r \in \mathbb{N}$  then  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is also a generator of the **olloid**  $\mathbb{F}_s^{k+r}$  of degree  $k+r$ .

*Proof.* Suppose  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is a generator of the **olloid**  $\mathbb{F}_s^k$  with continuous derivative on  $[1, s]$ . Then there exists a vector  $(v_1, v_2, \dots, v_s) \in \mathbb{F}_s^k$  such that  $v_i = g(i)$  for each  $1 \leq i \leq s$ , so that we can write

$$\sum_{i=1}^s g(i)^k = 1.$$

Let us assume to the contrary that there exists no  $r \in \mathbb{N}$  such that  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is a generator of the **olloid**  $\mathbb{F}_s^{k+r}$ . By applying the summation by parts, we obtain the inequality

$$(2.1) \quad \frac{1}{g(s)} \sum_{i=1}^s g(i)^{k+1} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt$$

by using the inequality

$$\sum_{i=1}^s g(i)^{k+1} < \sum_{i=1}^s g(i)^k = 1.$$

By applying summation by parts on the left side of (2.1) and using the contrary assumption, we obtain further the inequality

$$(2.2) \quad \frac{1}{g(s)^2} \sum_{i=1}^s g(i)^{k+2} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt.$$

By induction we can write the inequality as

$$\frac{1}{g(s)^r} \sum_{i=1}^s g(i)^{k+r} \geq 1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt - \dots - \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for any  $r \geq 2$  with  $r \in \mathbb{N}$ . Since  $g : \mathbb{N} \rightarrow \mathbb{R}^+$  is decreasing, it follows that

$$1 - \int_1^s \frac{g'(t)}{g(t)^2} dt - \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt - \dots - \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt > 1$$

and using the requirement

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \dots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

for  $r \in \mathbb{N}$ , we have the inequality

$$\begin{aligned} 1 &= \sum_{i=1}^s g(i)^k \\ &\geq \sum_{i=1}^s g(i)^{k+r} > 1 \end{aligned}$$

which is absurd. This completes the proof of the Lemma.  $\square$

Lemma 2.4 - albeit fundamental - is ultimately useful for our study of variants and possibly extensions of the Erdős-Moser equation. It can be seen as a tool for extending the solution of equations of the form

$$\sum_{i=1}^s g(i)^k = 1$$

for  $k > 1$  - under the presumption that it exists - to the solution of equations of the form

$$\sum_{i=1}^s g(i)^{k+r} = 1$$

for a fixed  $r \in \mathbb{N}$  under some special requirements of the generator  $g : \mathbb{N} \rightarrow \mathbb{R}$ .

### 3. Application to solutions of the generalized Erdős-Moser equation

In this section we apply the notion of the **olloid** to study solutions of the Erdős-Moser equation. We launch the following method as an outgrowth of Lemma 2.4.

**Theorem 3.1** (The generalized extension method). *Let  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  have continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$ . If the equation*

$$\sum_{i=1}^s h(i)^k = h(s+1)^k$$

for  $k > 1$  has a solution and there exist some  $r \in \mathbb{N}$  such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{h(i)}{h(s+1)}$$

for  $1 \leq i \leq s$ . Then the equation

$$\sum_{i=1}^s h(i)^{k+r} = h(s+1)^{k+r}$$

also has a solution.

*Proof.* Suppose the equation

$$(3.1) \quad \sum_{i=1}^s h(i)^k = h(s+1)^k$$

has a solution. Then equation (3.1) can be recast as

$$(3.2) \quad \sum_{i=1}^s \left( \frac{h(i)}{h(s+1)} \right)^k = 1$$

which can also be transformed into the sum

$$\sum_{i=1}^s g(i)^k = 1$$

with

$$g(i) := \frac{h(i)}{h(s+1)}.$$

The function

$$g(i) := \frac{h(i)}{h(s+1)}$$

for  $1 \leq i \leq s$  is decreasing and has continuous derivative on  $[1, s]$  since  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  have continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$ , so that if there exists some  $r \in \mathbb{N}$  such that

$$1 - \frac{1}{g(s)^r} > \int_1^s \frac{g'(t)}{g(t)^2} dt + \frac{1}{g(s)} \int_1^s \frac{g'(t)}{g(t)^2} dt + \cdots + \frac{1}{g(s)^{r-1}} \int_1^s \frac{g'(t)}{g(t)^2} dt$$

with

$$g(i) := \frac{h(i)}{h(s+1)}$$

for  $1 \leq i \leq s$ , then by appealing to Lemma 2.4 the equation

$$(3.3) \quad \sum_{i=1}^s g(i)^{k+r} = 1$$

also has a solution. We note that equation (3.3) can also be transformed to the equation

$$(3.4) \quad \sum_{i=1}^s \left( \frac{h(i)}{h(s+1)} \right)^{k+r} = 1$$

so that it has a solution. Since equation (3.4) can be recast as

$$\sum_{i=1}^s h(i)^{k+r} = h(s+1)^{k+r}$$

and the claim follows immediately.  $\square$

It is important to note that if the values of  $h$  on the positive integers is still a positive integer and have continuous derivative on  $[1, s]$  and decreasing on  $\mathbb{R}^+$ , then the integer solution of the more general Erdős-Moser equation

$$\sum_{i=1}^s h(i)^k = h(s+1)^k$$

can be extended to the integer solutions of the equation

$$\sum_{i=1}^s h(i)^{k+r} = h(s+1)^{k+r}$$

under the local condition of the normalized values of  $h$  on  $[1, s+1]$ . One could also examine the problem with the sequence  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  and ask if it is possible to take  $h$  to be an arithmetic progression. A similar question could be ask for sequences  $h : \mathbb{N} \rightarrow \mathbb{R}^+$  of general types. It is important to recognize that the tool we have developed only allows us to extend solutions of the general Erdős-Moser equation under a certain local condition of normalized generators of the **olloid**.

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