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Logistic equation on time scales

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A B S T R A C T

We propose a new dynamic logistic equation that, in contrast with the one available in the literature, preserves the non-negativity of its solutions, which is an essential property for biological meaning. We derive the exact solution of the proposed model and analyze its behavior on arbitrary time scales. Finally, we illustrate our results with concrete examples. Our key novelty is: the model's ability to maintain positivity across general time scales and its dynamic consistency with the classical logistic equation.

1. Introduction

Differential equations are widely recognized for their application in modeling the dynamics of a population over time. The logistic equation, pioneered in the 19th century by the Belgian mathematician Verhulst [1], extends the exponential growth model by introducing a maximum population capacity, known as the carrying capacity. Such equation is defined as

$$\dot{y} = ry\left(1 - \frac{y}{k}\right), \tag{1}$$

where $r > 0$ denotes the growth rate and $k > 0$ the carrying capacity. The exact solution of (1) is given by

$$y(t) = \frac{y(0)ke^{rt}}{k + y(0)(e^{rt} - 1)}, \tag{2}$$

where $y(0)$ denotes the initial value for the population y (see, e.g., [1]). Moreover, the behavior of such solution is well established: the total population increases (or decreases) progressively from $y(0)$ to the limit k, which is reached only when $t \to +\infty$. Although quite simple, the logistic equation remains a topic of ongoing interest and it is still nowadays widely explored by researchers worldwide (see, e.g., [2–4]).

The theory of time scales was introduced in 1988 by Stefan Hilger [5]. Such theory unifies and generalizes continuous and discrete time into a single framework [6]. Recently, this theory has been applied to several biological models [7–9]. However, to the best of our knowledge, there has been limited positive advancements in constructing the logistic equation on arbitrary time scales. In [10], Streipert presents the following analogue of the logistic growth model:

$$y^{\Delta} = ry^{\sigma}\left(1 - \frac{y}{k}\right), \tag{3}$$

where σ is the forward jump operator and y^{Δ} denotes the derivative of y on an arbitrary time scale $\mathbb{T}$ (see Definitions 1, 2, and 6 in Section 2). The initial population size $y(t_0) = y_0$ is considered to be positive at time $t_0 \in \mathbb{T}$, $r > 0$ is the growth rate, and $k > 0$ the carrying capacity. Eq. (3) yields the discrete-time analogue

$$y(t+1) - y(t) = ry(t+1)\left(1 - \frac{y(t)}{k}\right) \tag{4}$$

when the time scale is chosen to be the set of integer numbers, that is, $\mathbb{T} = \mathbb{Z}$. However, Eq. (4) is not consistent with the original continuous logistic equation, since it can produce negative values under particular conditions. Let us take a simple example: if we consider $y(0) = 2$, $r = 2$ and $k = 5$, we obtain $y(1) = -10$. This means that, even when we start with positive initial conditions, Eq. (4) can fail to guarantee the non-negativity of solutions, that is a crucial aspect of the original continuous-time logistic equation. It is the main goal of our work to present a dynamic logistic equation that provides a consistent formulation of the logistic growth equation for any arbitrary time scale.

In Section 2, crucial definitions and results from the time-scale calculus are recalled. In Section 3, we present our original results: a novel

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dynamic logistic equation, proving its consistency and deriving its exact solution. Moreover, we analyze the asymptotic behavior of the obtained solution. We end by presenting some applications for particular but important time scales. The new Eq. (6) we propose in Section 3 is fundamentally different and superior compared to Streipert's model (3) since it guarantees non-negativity of its solution and consistency with the classical logistic equation.

2. Fundamental concepts of time scales

In this section some basic concepts and results of time scales calculus are recalled. For more details, we refer the interested reader to the books [6,11].

Definition 1 (*See [6]*). A time scale $\mathbb{T}$ is any nonempty closed subset of the real numbers.

Definition 2 (*See [6]*). For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is given by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.

If $f : \mathbb{T} \to \mathbb{R}$, then we denote $f^\sigma = f \circ \sigma$. Using both backward and forward operators, any point $t \in \mathbb{T}$ can be defined as right-scattered (left-scattered) if $\sigma(t) > t$ ($\rho(t) < t$); or right-dense (left-dense) if $\sigma(t) = t$ ($\rho(t) = t$). Moreover, a point $t \in \mathbb{T}$ is isolated if it is left and right-scattered; or dense if it is left and right-dense.

Definition 3 (*See [6]*). The graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ is defined as $\mu(t) := \sigma(t) - t$.

Definition 4 (*See [6]*). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exists for all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 5 (*See [6]*). A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of regressive and rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Moreover, $f \in \mathcal{R}$ is called positively regressive, i.e., $f \in \mathcal{R}^+$, if $1 + \mu(t)f(t) > 0$ for all $t \in \mathbb{T}$.

For the following results, the subset $\mathbb{T}^\kappa$ is defined as follows: if $t \in \mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 6 (*See [6]*). Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then $f^\Delta(t)$ denotes the delta (or Hilger) derivative and we define it as the number (provided it exists) for which given any $\varepsilon > 0$ there is a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. Moreover, if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$, we say that f is delta differentiable.

Remark 1. If $\mathbb{T} = \mathbb{R}$, then $x^\sigma(t) = x(t)$ and $x^\Delta(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^\sigma(t) = x(t+1)$ and $x^\Delta(t) = x(t+1) - x(t)$.

Definition 7 (*See [6]*). Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then $y^\Delta = f(t, y, y^\sigma)$ is called a first order dynamic equation. Additionally, given $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$, the problem

$$y^\Delta = f(t, y, y^\sigma), \quad y(t_0) = y_0, \tag{5}$$

is called an initial value problem (IVP).

Theorem 1 (*See [6]*). *Let $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$. Then the regressive IVP problem of the form $y^\Delta = p(t)y$, $y(t_0) = 1$, has the exponential function as its unique solution, denoted by $e_p(\cdot, t_0)$.*

Table 1
Exponential function on different time scales.

	$\mathbb{T}$	$\mathbb{R}$	$h\mathbb{Z}$	$q^{\mathbb{N}_0}$
Exponential function	$e_p(t, t_0)$	$e^{p(t-t_0)}$	$(1+ph)^{\frac{t-t_0}{h}}$	$\prod_{s\in[t_0,t)}(1+(q-1)ps)$

Table 1 gives some examples of the exponential function $e_p(\cdot, t_0)$ on specific time scales.

According to Table 1, it is clear that, under certain circumstances, the positivity property inherent to the classical exponential function does not hold. For instance, when $\mathbb{T} = h\mathbb{Z}$ for certain values of $p \in \mathcal{R}$, the exponential function may be negative. This property is restored if we choose $p \in \mathcal{R}^+$. Thus, the next result follows.

Theorem 2 (*See [6]*). *If $p \in \mathcal{R}^+$ and $t_0 \in \mathbb{T}$, then $e_p(t, t_0) > 0$, for all $t \in \mathbb{T}$.*

Theorem 3 summarizes some properties of the exponential function that are needed in the sequel. First we recall the definition of "circle minus" subtraction.

Definition 8 (*See [11]*). The "circle minus" subtraction $\ominus$ on $\mathcal{R}$ is defined by $p \ominus q = \frac{p-q}{1+\mu q}$. In the case $p = 0$ we use the notation $\ominus q$ instead of $0 \ominus q$, that is, $\ominus q = \frac{-q}{1+\mu q}$.

Theorem 3 (*See [6]*). *If $p \in \mathcal{R}$, then*

- $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
- $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
- $e_p(t, s)e_p(s, r) = e_p(t, r)$.

Lemma 1. *Suppose $p \in \mathcal{R}^+$. Then, $e_p(t, s)$ is strictly increasing.*

Proof. We know that $e_p^\Delta(t, s) = p\, e_p(t, s)$. By hypothesis, $p > 0$ and $e_p(t, s) > 0$. Thus, it is clear that $e_p^\Delta(t, s) > 0$, which means that $e_p(t, s)$ is a strictly increasing function. □

Theorem 4 (*See [6]*). *If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(t, c)\Delta(t) = e_p(b, c) - e_p(a, c).$$

The following result is crucial to obtain the solution to a linear, nonhomogeneous, first order dynamic equation.

Theorem 5 (*Variation of Constants [11]*). *Suppose $p \in \mathcal{R}$ and $f \in C_{rd}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the IVP $y^\Delta = -p(t)y^\sigma + f(t)$, $y(t_0) = t_0$, is given by*

$$y(t) = e_{\ominus p}(t, t_0)\left[y_0 + \int_{t_0}^t e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau\right].$$

3. Dynamic logistic equation

In this section we introduce a new dynamic logistic equation and derive its exact solution. In the particular case of $\mathbb{T} = \mathbb{Z}$, this provides a novel discrete time equation that is consistent with the classical one (1). We analyze the asymptotic behavior of the dynamic equation and end our discussion with some illustrative examples. Consider the dynamic logistic equation defined as

$$y^\Delta = ry\left(1 - \frac{y^\sigma}{k}\right), \tag{6}$$

with $y(t_0) = y_0 > 0$. Here, $r > 0$ denotes the growth rate and $k > 0$ the carrying capacity.

Theorem 6. *The exact solution of the dynamic equation (6) is given by*

$$y(t) = \frac{k\, y_0\, e_r(t,t_0)}{k + y_0(e_r(t,t_0) - 1)}. \tag{7}$$

Proof. Let us define $z = \frac{1}{y}$. Then, $z^{\Delta} = -\frac{y^{\Delta}}{yy^{\sigma}} = \frac{-ry\left(1-\frac{y^{\sigma}}{k}\right)}{yy^{\sigma}} = -\frac{r}{y^{\sigma}} + \frac{r}{k}$. Moreover, by definition of the forward jump operator, we obtain $z^{\Delta} = -rz^{\sigma} + \frac{r}{k}$, which is a linear first order dynamic equation whose solution follows from Theorem 5 as

$$z(t) = e_{\ominus r}(\tau, t_0)\left[z_0 + \frac{1}{k}\int_{t_0}^{t} e_{\ominus r}(t_0,\tau)\, r\,\Delta\tau\right].$$

Evaluating the integral, it follows from Theorems 3 and 4 that

$$\frac{1}{k}\int_{t_0}^{t} e_{\ominus r}(t_0,\tau)\, r\,\Delta\tau = \frac{1}{k}\int_{t_0}^{t} e_{r}(\tau,t_0)\, r\,\Delta\tau = \frac{1}{k}\left(e_r(t,t_0) - 1\right).$$

Thus,

$$\begin{aligned} y(t) = \frac{1}{z(t)} &= \frac{1}{e_{\ominus r}(t,t_0)\left[\frac{1}{y_0} + \frac{1}{k}(e_r(t,t_0)-1)\right]} \\ &= \frac{e_r(t,t_0)}{\frac{1}{y_0} + \frac{1}{k}(e_r(t,t_0)-1)} = \frac{k\, y_0\, e_r(t,t_0)}{k + y_0(e_r(t,t_0)-1)}. \end{aligned}$$

The result is proved. □

Remark 2. The reader is invited to compare the exact solution (7) given in Theorem 6 with the classical one (1). We can see that the form (and meaning) of the solution is the same in any time scale, just the exponential taking different expressions (see the explicit form of the exponential function on different time scales in Table 1). For more on the meaning of the solution see Remark 3.

Proposition 1. *The exact solution (7) remain non-negative for all $t \in \mathbb{T}$.*

Proof. By definition, r, k, $y_0 > 0$. Then, $r \in \mathcal{R}^+$. Thus, from Theorem 2, $e_r(t,t_0) > 0$ for all $t \in \mathbb{T}$. Moreover, as $e_0(t,t_0) \equiv 1$ and, since $e_r(t,t_0)$ is strictly increasing (Lemma 1), it is clear that $e_r(t,t_0) - 1 > 0$. Thus, the result follows. □

It is clear that 0 and k are equilibrium points of (6): if $y(t) = 0$ or $y(t) = k$, then $y^{\Delta} = 0$.

Theorem 7. *Suppose $\mathbb{T}$ is unbounded from above. All solutions $y(t)$ of (6) converge to the equilibrium k.*

Proof. We first note that the exact solution (7) can be rewritten as

$$y(t) = \frac{1}{e_{\ominus r}(t,t_0)\left(\frac{1}{y_0} - \frac{1}{k}\right) + \frac{1}{k}}.$$

Since $r > 0$ for all $t \in \mathbb{T}$, then $r \in \mathcal{R}^+$. Thus, by Lemma 1, $e_r(t,t_0)$ is strictly increasing. As $\mathbb{T}$ is unbounded from above, this means that $\lim_{t\to\infty} e_r(t,t_0) = +\infty$ and $\lim_{t\to\infty} e_{\ominus r}(t,t_0) = 0$. Thus, $\lim_{t\to\infty} y(t) = k$, which ends our proof. □

Remark 3. The solution (7) shows that initially (small t), the population $y(t)$ grows approximately exponentially, as the denominator is close to k and the exponential dominates. According to Theorem 7, as $t \to \infty$, the exponential term $e_r(t,t_0)$ becomes very large, so the solution approaches k, which is the *carrying capacity* of the environment. The form (7) of the exact solution shows how the growth is limited by the carrying capacity k, preventing indefinite exponential growth. The term $k + y_0(e_r(t,t_0) - 1)$ of solution (7) reflects the effect of limited resources, slowing growth as $y(t)$ approaches k.

As examples of applications of Eq. (6), we introduce the discrete and the quantum logistic growth model. The discrete version is given by

$$\frac{y_{n+1} - y_n}{h} = r y_n\left(1 - \frac{y_{n+1}}{k}\right), \tag{8}$$

which is equivalent to

$$y_{n+1} = \frac{k y_n(1+hr)}{k + hr y_n}. \tag{9}$$

It is trivial to see from (9) that the solutions are non-negative over time. Moreover, the following corollary is a direct consequence of Theorem 6.

Corollary 1. *The exact solution of the discrete-time logistic equation (8) is given by*

$$y_n = \frac{k y_0(1+hr)^n}{k + y_0[(1+hr)^n - 1]}. \tag{10}$$

In Fig. 1, we compare particular solutions between the discrete-time model described by Streipert in [10] and our version (8). Despite both models converging to the equilibrium k, Eq. (4) of Streipert exhibits the possibility of reaching negative values, even with a positive initial condition $y(0)$. Furthermore, in certain cases, Eq. (4) exhibits a periodic behavior (Fig. 2), a phenomenon that is not coherent with the classical logistic equation. In contrast, our model has a behavior that is always coherent with the continuous/classical model (1).

It is worth to note that both Figs. 1 and 2 show a clear discrepancy between Streipert's model (3) and the expected logistic behavior. In contrast, our model shows consistence with the expected biological behavior.

The quantum analogue [12] of our Eq. (6) is

$$\frac{y(qt) - y(t)}{(q-1)t} = r y(t)\left(1 - \frac{y(qt)}{k}\right), \tag{11}$$

which is equivalent to

$$y(qt) = \frac{k y(t)(1 + r(q-1)t)}{k + r y(t)(q-1)t}.$$

Remark 4. In the sequel, we use the notation $x(t_n)$, where $n \in \mathbb{N}_0$. By definition, $t_1 = qt_0$, $t_2 = qt_1 = q^2 t_0$, …, i.e., $t_n = q^n t_0$.

The exact solution of Eq. (11) follows from Theorem 6.

Corollary 2. *The exact solution of the quantum-time logistic equation (11) is given by*

$$y(t_n) = \frac{k\, y_0 \prod_{s\in[t_0,t)}(1 + (q-1)rs)}{k + y_0\left[\left(\prod_{s\in[t_0,t)} 1 + (q-1)rs\right) - 1\right]}. \tag{12}$$

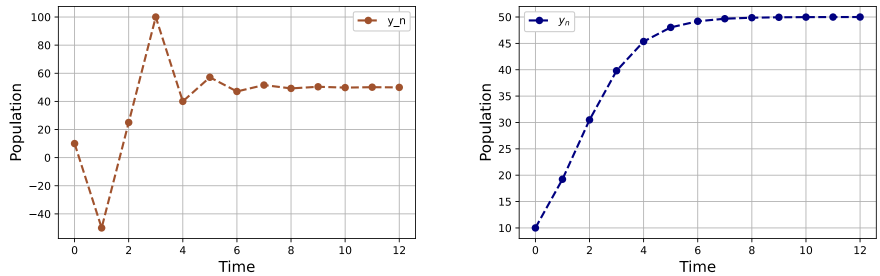
Again, since $q > 1$, it is clear that solution (12) of (11) preserves the non-negativity property.

In Fig. 3, we compare Streipert's [10] quantum-time logistic equation that follows from (3) with our proposed model (11). In this time-scale it is also evident that the behavior of Streipert's equation diverges significantly from the classical logistic growth model while our model is dynamical coherent.

Similarly to the discrete-time case, Fig. 3 shows that there is also a clear discrepancy between Streipert's model and the expected logistic behavior in the quantum case. Such lack of consistence is completely solved by our model.

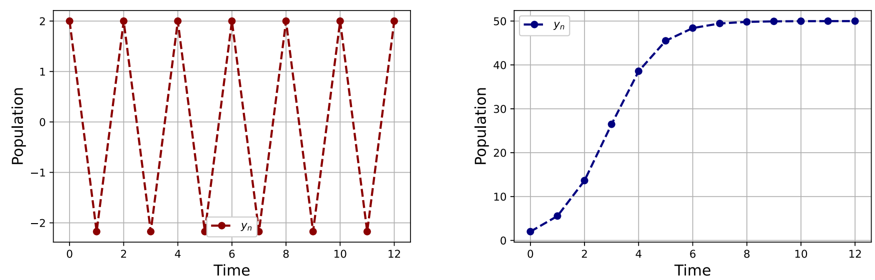
4. Conclusion

We introduced a novel formulation of the logistic equation on arbitrary time scales. Our dynamic model preserves the non-negativity of solutions – a crucial property for biologically meaningful interpretations and population dynamics. Furthermore, we derive the exact solution of the proposed equation and explore its behavior for continuous, discrete, and quantum time domains, offering valuable analytical insights and concrete comparisons with prior discrete analogues in the literature. The presented model is new and a mathematically robust formulation of the logistic equation that overcomes limitations in prior models on time scales.



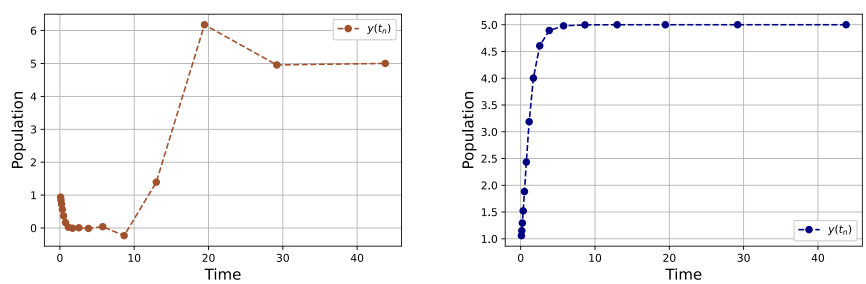
(a) Solution of Streipert's [10] discrete-time model (4) (b) Solution of our discrete-time model (8)

Fig. 1. Population growth with $y_0 = 10$, $k = 50$, $r = 1.5$ and $h = 1$.



(a) Solution of Streipert's [10] discrete-time model (4) (b) Solution of our discrete-time model (8)

Fig. 2. Population growth with $y_0 = 2$, $k = 50$, $r = 2$ and $h = 1$.



(a) Solution of equation (3) when $\mathbb{T} = q^{\mathbb{N}_0}$ (b) Solution of the quantum-time model (11)

Fig. 3. Population growth with $y_0 = 1$, $k = 5$, $r = 1.5$ and $q = 1.5$.

As noted in [13] in the context of the discrete-time SIR model, while the forward jump operator is not visible in the continuous case, in discrete-time its position is fundamental to guarantee biological significance. We claim that the results we have obtained here, for an arbitrary time scale, can be extended to other types of nonlinear dynamic equations on time scales. This is currently under investigation for the SIR model and will be presented elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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