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DOI: 10.14293/111.000/000039.v1
Preprint first posted online: 13 July 2022
Keywords: electrostatic energy, Coulomb, electrons, minimal

ON A RELATED THOMPSON PROBLEM IN $\mathbb{R}^k$

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Abstract. In this paper we study the global electrostatic energy behaviour of mutually repelling charged electrons on the surface of a unit-radius sphere. Using the method of compression, we show that the total electrostatic energy $U_k(N)$ of N mutually repelling particles on a sphere of unit radius in $\mathbb{R}^k$ satisfies the lower bound

$$U_k(N) \gg_{\epsilon} \frac{N^2}{\sqrt{k}}.$$

1. Introduction

According to Coulomb's law the electrostatic potential energy between any two pair electrons with equal charges $e = e_i = e_j$ on the surface of a sphere with respective position vectors $\vec{r}_i$ and $\vec{r}_j$ is given as

$$U_{ij} = K_e \frac{e_i e_j}{\vec{r}_{ij}}$$

where K_e is Coulomb's constant and $\vec{r}_{ij} = ||\vec{r}_i - \vec{r}_j||$ is the distance between the pair of charged electron. By making the assignment $e = e_i = e_j = 1$ and $K_e = 1$, it can be seen that the electrostatic potential energy reduces to

$$U_{ij} = \frac{1}{\vec{r}_{ij}}.$$

To this end, the total electrostatic energy interaction among N charged electrons is given by the sum

$$U(N) = \sum_{1 \leq i < j \leq N} \frac{1}{||\vec{r}_i - \vec{r}_j||}.$$

The Thompson Problem is a conundrum that asks for the arrangement of any number of electrons on the surface of a unit-radius sphere with the lowest possible global electrostatic potential energy. In other words, when the lowest total electrostatic potential energy exists between pairs of electrons on the sphere's surface, the problem will be solved. In order to do this, it is worthwhile to examine and minimize the overall behavior of the total energy for sufficiently large number of electrons. Only a few particular cases of the problem have been addressed, and the actual problem has not made much progress. Since a single electron is not affected by any external forces, the case of a single electron presents a trivial problem. The best solution for the two electron scenario is to arrange the electrons on the sphere so

Date: July 13, 2022.
2000 *Mathematics Subject Classification.* Primary 54C40, 14E20; Secondary 46E25, 20C20.
Key words and phrases. minimal; electrostatic energy; Coulomb; electrons.

that they are both **antipodal**, or that the line connecting them runs through the center of the sphere. This results in a minimum total electrostatic potential energy of $U(2) = \frac{1}{2}$. When there are more than three electrons present, the situation ceases to be trivial. Electrons forming an **equilateral** triangle on a great circle of a sphere have been shown to be the best way to solve the Thompson issue for the case of three electrons [2]. On the other hand, it is known that the vertices of a regular **tetrahedron** contain the four electron case. Massive computation was used to rigorously arrive at the best answer for the five electron scenario (see [5]), where it is demonstrated that the vertices of a triangular **dipyramid** must hold the electrons. The discovery that the six electron case exists on an **octahedron** can be found in [7], and that the twelve electron case holds for electrons resting on the vertices of an **icosahedron** in [1]. There are many more known special situations, but it has not yet been possible to arrange all sufficiently large N electrons on a surface of a sphere in a way that has the lowest possible overall electrostatic potential energy. The primary way of attack for the Thompson problem has been the use of algorithmic and local optimization techniques to the energy function $U(N)$ [3], [4],[8]. The issue with the general total energy function of the form

$$\sum_{i<j} f(||\vec{r}_i - \vec{r}_j||)$$

where f is a decreasing real-valued function, can also be taken into consideration. The problem can also be studied for spheres in higher dimensions $\mathbb{R}^k$ for $k \geq 3$. It is an unsolved and a related problem to find the minimum Coulomb potential

$$U(N) = \sum_{i<j\leq N} \frac{1}{||\vec{r}_i - \vec{r}_j||}.$$

In this paper, we apply the method of compression [6] to study the global electrostatic potential energy behaviour of all sufficiently large number of charged electrons repelling each other on the surface of a sphere; in particular, we obtain the result

Theorem 1.1. *Let $U_k(N)$ denotes the total electrostatic energy of N mutually repelling particles on a sphere of unit radius in $\mathbb{R}^k$. Then we have*

$$U_k(N) \gg_{\epsilon} \frac{N^2}{\sqrt{k}}.$$

1.1. **Notations and conventions.** Through out this paper, we will assume that N is sufficiently large. We write $f(s) \gg g(s)$ if there there exists a constant $c > 0$ such that $f(s) \geq c|g(s)|$ for all s sufficiently large. If the constant depends of some variable, say t, then we denote the inequality by $f(s) \gg_t g(s)$. We write $f(s) = o(g(s))$ if the limits holds $\lim\limits_{s \longrightarrow \infty} \frac{f(s)}{g(s)} = 0$.

2. **Preliminaries and background**

Definition 2.1. By the compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed on $\mathbb{R}^n$ we mean the map $\mathbb{V} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right)$$

for $n \geq 2$ and with $x_i \neq x_j$ for $i \neq j$ and $x_i \neq 0$ for all $i = 1, \ldots, n$.

Remark 2.2. The notion of compression is in some way the process of re scaling points in $\mathbb{R}^n$ for $n \geq 2$. Thus it is important to notice that a compression roughly speaking pushes points very close to the origin away from the origin by certain scale and similarly draws points away from the origin close to the origin.

Proposition 2.1. *A compression of scale* $1 \geq m > 0$ *with* $\mathbb{V}_m : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a bijective map.*

Proof. Suppose $\mathbb{V}_m[(x_1, x_2, \ldots, x_n)] = \mathbb{V}_m[(y_1, y_2, \ldots, y_n)]$, then it follows that

$$\left(\frac{m}{x_1}, \frac{m}{x_2}, \ldots, \frac{m}{x_n}\right) = \left(\frac{m}{y_1}, \frac{m}{y_2}, \ldots, \frac{m}{y_n}\right).$$

It follows that $x_i = y_i$ for each $i = 1, 2, \ldots, n$. Surjectivity follows by definition of the map. Thus the map is bijective. □

2.1. **The mass of compression.** In this section we recall the notion of the mass of compression on points in space and study the associated statistics.

Definition 2.3. By the mass of a compression of scale $m > 0$ ($m \in \mathbb{R}$) fixed, we mean the map $\mathcal{M} : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) = \sum_{i=1}^{n} \frac{m}{x_i}.$$

It is important to notice that the condition $x_i \neq x_j$ for $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is not only a quantifier but a requirement; otherwise, the statement for the mass of compression will be flawed completely. To wit, suppose we take $x_1 = x_2 = \cdots = x_n$, then it will follows that $\mathrm{Inf}(x_j) = \mathrm{Sup}(x_j)$, in which case the mass of compression of scale m satisfies

$$m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) - k} \leq \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \ldots, x_n)]) \leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}$$

and it is easy to notice that this inequality is absurd. By extension one could also try to equalize the sub-sequence on the bases of assigning the supremum and the Infimum and obtain an estimate but that would also contradict the mass of compression inequality after a slight reassignment of the sub-sequence. Thus it is important for the estimate to make any good sense to ensure that any tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ must satisfy $x_i \neq x_j$ for all $1 \leq i, j \leq n$. Hence in this paper this condition will be highly extolled. In situations where it is not mentioned, it will be assumed that the tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is such that $x_i \leq x_j$ for $1 \leq i, j \leq n$.

Lemma 2.4. *The estimate holds*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right)$$

where $\gamma = 0.5772\cdots$.

Remark 2.5. Next we prove upper and lower bounding the mass of the compression of scale $m > 0$.

Proposition 2.2. *Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for each $1 \leq i \leq n$ and $x_i \neq x_j$ for $i \neq j$, then the estimates holds*

$$m \log \left(1 - \frac{n-1}{\sup(x_j)}\right)^{-1} \ll \mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) \ll m \log \left(1 + \frac{n-1}{\mathrm{Inf}(x_j)}\right)$$

for $n \geq 2$.

Proof. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ for $n \geq 2$ with $x_j \neq 0$. Then it follows that

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\leq m \sum_{k=0}^{n-1} \frac{1}{\mathrm{Inf}(x_j) + k}\end{aligned}$$

and the upper estimate follows by the estimate for this sum. The lower estimate also follows by noting the lower bound

$$\begin{aligned}\mathcal{M}(\mathbb{V}_m[(x_1, x_2, \dots, x_n)]) &= m \sum_{j=1}^{n} \frac{1}{x_j} \\ &\geq m \sum_{k=0}^{n-1} \frac{1}{\sup(x_j) - k}.\end{aligned}$$

□

Definition 2.6. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, 2 \dots, n$. Then by the gap of compression of scale $m > 0$, denoted $\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)]$, we mean the expression

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)] = \left|\left|\left(x_1 - \frac{m}{x_1}, x_2 - \frac{m}{x_2}, \dots, x_n - \frac{m}{x_n}\right)\right|\right|$$

Definition 2.7. Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with $x_i \neq 0$ for all $1 \leq i \leq n$. Then by the ball induced by $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ under compression of scale $m > 0$, denoted $\mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)]}[(x_1, x_2, \dots, x_n)]$ we mean the inequality

$$\left|\left|\vec{y} - \frac{1}{2}\left(x_1 + \frac{m}{x_1}, x_2 + \frac{m}{x_2}, \dots, x_n + \frac{m}{x_n}\right)\right|\right| < \frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)].$$

A point $\vec{z} = (z_1, z_2, \dots, z_n) \in \mathcal{B}_{\frac{1}{2}\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)]}[(x_1, x_2, \dots, x_n)]$ if it satisfies the inequality. We call the ball the circle induced by points under compression if we take the dimension of the underlying space to be $n = 2$.

Remark 2.8. In the geometry of balls under compression of scale $m > 0$, we will assume implicitly that $1 \geq m > 0$. The circle induced by points under compression is the ball induced on points when we take $n = 2$.

Proposition 2.3. *Let $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ for $n \geq 2$, then we have*

$$\mathcal{G} \circ \mathbb{V}_m[(x_1, x_2, \dots, x_n)]^2 = \mathcal{M} \circ \mathbb{V}_1\left[\left(\frac{1}{x_1^2}, \dots, \frac{1}{x_n^2}\right)\right] + m^2\mathcal{M} \circ \mathbb{V}_1[(x_1^2, \dots, x_n^2)] - 2mn.$$

In particular, we have the estimate

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2=\mathcal{M}\circ\mathbb{V}_1\left[\left(\frac{1}{x_1^2},\ldots,\frac{1}{x_n^2}\right)\right]-2mn+O\left(m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]\right)$$

for $\vec{x}\in\mathbb{N}^n$*, where* $m^2\mathcal{M}\circ\mathbb{V}_1[(x_1^2,\ldots,x_n^2)]$ *is the error term in this case.*

Lemma 2.9 (Compression estimate)**.** *Let* $(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ *for* $n\geq 2$ *and* $x_i\neq x_j$ *for* $i\neq j$*, then we have*

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\ll n\mathrm{sup}(x_j^2)+m^2\log\left(1+\frac{n-1}{\mathrm{Inf}(x_j)^2}\right)-2mn$$

and

$$\mathcal{G}\circ\mathbb{V}_m[(x_1,x_2,\ldots,x_n)]^2\gg n\mathrm{Inf}(x_j^2)+m^2\log\left(1-\frac{n-1}{\mathrm{sup}(x_j^2)}\right)^{-1}-2mn.$$

Theorem 2.10. *Let* $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ *with* $z_i\neq z_j$ *for all* $1\leq i<j\leq n$*. Then* $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ *if and only if*

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

Proof. Let $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ for $\vec{z}=(z_1,z_2,\ldots,z_n)\in\mathbb{N}^n$ with $z_i\neq z_j$ for all $1\leq i<j\leq n$, then it follows that $||\vec{y}||>||\vec{z}||$. Suppose on the contrary that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{y}],$$

then it follows that $||\vec{y}||\leq||\vec{z}||$, which is absurd. Conversely, suppose

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{y}]$$

then it follows from Proposition 2.3 that $||\vec{z}||<||\vec{y}||$. It follows that

$$\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|<\left|\left|\vec{y}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right|$$
$$=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}].$$

This certainly implies $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ and the proof of the theorem is complete. □

Theorem 2.11. *Let* $\vec{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{N}^n$ *with* $x_i\neq x_j$ *for all* $1\leq i<j\leq n$*. If* $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ *then*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\subseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ and suppose for the sake of contradiction that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\nsubseteq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there must exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]$ such that $\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. It follows from Theorem 2.10 that

$$\mathcal{G}\circ\mathbb{V}_m[\vec{z}]\geq\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\begin{aligned}\mathcal{G}\circ\mathbb{V}_m[\vec{y}] &> \mathcal{G}\circ\mathbb{V}_m[\vec{z}]\\ &\geq \mathcal{G}\circ\mathbb{V}_m[\vec{x}]\\ &> \mathcal{G}\circ\mathbb{V}_m[\vec{y}]\end{aligned}$$

which is absurd, thereby ending the proof. □

Remark 2.12. Theorem 2.11 tells us that points confined in certain balls induced under compression should by necessity have their induced ball under compression covered by these balls in which they are contained.

2.2. **Admissible points of balls induced under compression.** We launch the notion of admissible points of balls induced by points under compression. We study this notion in depth and explore some possible connections.

Definition 2.13. Let $\vec{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$ with $y_i\neq y_j$ for all $1\leq i<j\leq n$. Then $\vec{y}$ is said to be an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ if

$$\left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|=\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

Remark 2.14. It is important to notice that the notion of admissible points of balls induced by points under compression encompasses points on the ball. These points in geometrical terms basically sit on the outer of the induced ball.

Theorem 2.15. *The point $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ is admissible if and only if*

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}]=\mathcal{G}\circ\mathbb{V}_m[\vec{x}]$.

Proof. First let $\vec{y}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ be admissible and suppose on the contrary that

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]\neq\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}].$$

Then there exist some $\vec{z}\in\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$ such that

$$\vec{z}\notin\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}].$$

Applying Theorem 2.10, we obtain the inequality

$$\mathcal{G}\circ\mathbb{V}_m[\vec{y}]\leq\mathcal{G}\circ\mathbb{V}_m[\vec{z}]<\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows from Proposition 2.3 that $||\vec{x}||<||\vec{y}||$ or $||\vec{y}||<||\vec{x}||$. By joining this points to the origin by a straight line, this contradicts the fact that the point $\vec{y}$ is an admissible point of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$. The latter equality follows from assertion that two balls are indistinguishable. Conversely, suppose

$$\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{y}]}[\vec{y}]=\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}]$$

and $\mathcal{G}\circ\mathbb{V}_m[\vec{y}] = \mathcal{G}\circ\mathbb{V}_m[\vec{x}]$. Then it follows that the point $\vec{y}$ lives on the outer of the indistinguishable balls and must satisfy the inequality

$$\left|\left|\vec{z}-\frac{1}{2}\left(y_1+\frac{m}{y_1},\ldots,y_n+\frac{m}{y_n}\right)\right|\right| = \left|\left|\vec{z}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|$$
$$= \frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}].$$

It follows that

$$\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}] = \left|\left|\vec{y}-\frac{1}{2}\left(x_1+\frac{m}{x_1},\ldots,x_n+\frac{m}{x_n}\right)\right|\right|$$

and $\vec{y}$ is indeed admissible, thereby ending the proof. □

Remark 2.16. We note that we can replace the set $\mathbb{N}^n$ used in our construction with $\mathbb{R}^n$ at the compromise of imposing the restrictions $\vec{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ such that $x_i>1$ for all $1\leq i\leq n$ and $x_i\neq x_j$ for $i\neq j$. The following construction in our next result in the sequel employs this flexibility.

3. **The lower bound**

Theorem 3.1. *Let $U_k(N)$ denotes the total electrostatic energy of N mutually repelling particles on a sphere of unit radius in $\mathbb{R}^k$. Then we have*

$$U_k(N) \gg_{\epsilon} \frac{N^2}{\sqrt{k}}.$$

Proof. Pick arbitrarily a point $(x_1,x_2,\ldots,x_k)=\vec{x}_j\in\mathbb{R}^k$ with $x_i>1$ for $1\leq i\leq k$ and $x_i\neq x_l$ for $i\neq l$ such that $\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]=2$. This is achieved by choosing points $\vec{x}_j$ with the property that

$$\max_{1\leq j\leq \frac{N}{2}}\sup(x_{j_i})_{i=1}^{k} = 2+\epsilon$$

for some small $\epsilon>0$. This ensures the ball induced under compression is of radius

$$r=\frac{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]}{2}=1$$

and of center the midpoint of the compression line, joining the points $\vec{x}$ and $\mathbb{V}_m[\vec{x}]$. It is easy to see that the origin is not the center of the ball and points in the ball are not symmetric to the origin. The ball can always be translated so that the origin is at the center of the ball. Next we apply the compression of fixed scale $m\leq 1$, given by $\mathbb{V}_m[\vec{x}_j]$ and construct the ball induced by the compression given by

$$K := \mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}]}[\vec{x}_j]$$

with radius $\frac{(\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j])}{2}=1$. Now we choose $\frac{N}{2}$ admissible points $\vec{x}_j$ including their corresponding $\frac{N}{2}$ image points which are also admissible points of the ball $\mathcal{B}_{\frac{1}{2}\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]}[\vec{x}_j]$ constructed. Let us denotes the set of all admissible points of the ball - points on the surface of the sphere - with $K_{admissible}$. It follows that the total electrostatic energy of any mutually repelling particles on the sphere is given by

$$\begin{aligned}
U_k(N) &= \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_i,\vec{x}_j\in K_{admissible}\\ 1\leq t<j\leq N}} \frac{1}{||\vec{x}_t-\vec{x}_j||}\\
&\geq \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_j\in K_{admissible}\\ 1\leq j\leq \frac{N}{2}\\ \mathbb{V}_m[\vec{x}_j]=\vec{x}_t}} \frac{1}{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]} + \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_i,\vec{x}_j\in K_{admissible}\\ 1\leq t<j\leq \frac{N}{2}\\ \mathbb{V}_m[\vec{x}_j]\neq\vec{x}_t}} \frac{1}{||\vec{x}_t-\vec{x}_j||}\\
&> \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_j\in K_{admissible}\\ 1\leq j\leq \frac{N}{2}\\ \mathbb{V}_m[\vec{x}_j]=\vec{x}_t}} \frac{1}{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]} + \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_i,\vec{x}_j\in K_{admissible}\\ 1\leq t<j\leq \frac{N}{2}\\ \mathbb{V}_m[\vec{x}_j]\neq\vec{x}_t}} \frac{1}{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]}\\
&\gg \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \vec{x}_i,\vec{x}_j\in K_{admissible}\\ 1\leq t<j\leq \frac{N}{2}\\ \mathbb{V}_m[\vec{x}_j]\neq\vec{x}_t}} \frac{1}{\mathcal{G}\circ\mathbb{V}_m[\vec{x}_j]}\\
&\gg \sum_{\substack{(\vec{x}_t,\vec{x}_j)\\ \max_{1\leq j\leq \frac{N}{2}}\sup(x_{j_i})_{i=1}^{k}=2+\epsilon\\ 1\leq i<j\leq \frac{N}{2}}} \frac{1}{\sup(x_{j_i})_{i=1}^{k}\sqrt{k}}\\
&\gg \sum_{\substack{(\vec{x}_i,\vec{x}_j)\\ \max_{1\leq j\leq \frac{N}{2}}\sup(x_{j_i})_{i=1}^{k}=2+\epsilon\\ 1\leq i<j\leq \frac{N}{2}}} \frac{1}{\max_{1\leq j\leq \frac{N}{2}}\sup(x_{j_i})_{i=1}^{k}\sqrt{k}}\\
&= \frac{1}{(2+\epsilon)\sqrt{k}} \sum_{\substack{(\vec{x}_i,\vec{x}_j)\\ 1\leq i<j\leq \frac{N}{2}}} 1\\
&= \frac{1}{(2+\epsilon)\sqrt{k}} \times \binom{\frac{N}{2}}{2}\\
&\gg \frac{1}{(2+\epsilon)\sqrt{k}} \times N^2
\end{aligned}$$

and the lower bound follows. □

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