THE NECESSARY AND SUFFICIENT CONDITION FOR THE δ - POROSITY OF THE CONVERGENCE FIELD OF REGULAR MATRIX TRANSFORMATIONS

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ABSTRACT: *It has been shown by some researchers that the convergence field* $F(A)$ *of regular matrix transformation is a very porous set in the space* S *of all sequence of real or complex numbers while in [3], it have been proven to be σ-porous set in the linear metric space* $S(A)$ *endowed with Fréchet metric. Also, the usefulness of the well-known theorem on discontinuity points of function of the first bare class has been presented. Thus in this paper, we provided the necessary and sufficient conditions under which the convergence field of a various matrix transformation* $F(A)$ *is σ-porous.*

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1.0 Introduction

Matrix transformations constitute a cornerstone of summability theory, providing a robust mechanism for extending classical convergence to divergent sequences through linear operations. The convergence field F_A of a regular matrix $A = (a_{nk})$ defined as the set of all sequences transformed by $A = (a_{nk})$ into convergent sequences serves as a critical object for analyzing the efficacy and limitations of summation methods. Categorical properties such as meagerness of these domains has historically been the major area of focused while finer geometric classifications via σ-porosity which quantifies "topological thinness" through the existence of distributional holes in the space remain underdeveloped regardless their ability to distinguish between degrees of structural sparseness in infinite dimensional settings. This gap is particularly salient given the proliferation of matrix based summation techniques (e.g., Cesàro, Euler, Nörlund) in Fourier analysis, sequence spaces, and functional approximation.

Several researchers has studied the convergence field F_A of these matrix transformation. For instance, Šalát (1976), proved the usefulness of the theorem of the discontinuity points of functions of the first Baire while Kostyrko (2004) discussed some properties of the convergence field of regular matrix transformation of bounded sequences of real numbers. Visnyai (2006) proved the generalization of Steinhaus theorem for sequences of a branch space and showed that the result of Salat (1976) and Kostyrko (2004) can be generalized for a space of sequences of element of a Banach space $(X,\|.\|)$. Later Kostyrko (2008) showed that the convergence field $F(A)$ of a regular matrix transformation is a δ-porous set in the metric space $S(A)$ endowed with Fréchet metric. In his own view, Letavaj (2012) showed that the convergence field $F(A)$ of a regular matrix transformation is not only δ-porous but a very porous set in S while Siddiqui and Nkuno (2020) proved that the convergence field $F(A)$ of a regular matrix transformation is dense and closed in $S(A)$ if the given matrix is regular and porous if the matrix is irregular.

Thus in this paper, present the necessary and sufficient conditions for the convergence field $F(A)$ of regular matrix transformation δ-porous subset of the space $S(A)$, the application domain of the matrix $A = (a_{nk})$

2.0 Preliminaries

Definition 2.1 (Šalát, 1976): Let (Y,ρ) be a metric space and let Z be subset of X. Define
$\lambda(Z,x,r) = \sup\{t>0 : \exists\, z \in B(x,r); B(z,t)\subset B(x,r) \text{ and } B(z,t)\cap Z=\emptyset\}$.
The upper porosity of the set Z at the point x and the upper porosity of Z are defined as

$$\bar{p}(Z,x) = \lim_{r\to 0}\sup \frac{\lambda(Z,\ x,\ r)}{r} \text{ and } \bar{p}(Z) = \inf\{\bar{p}(Z,x): x \in Z\}.$$

We say that Z is porous if $\bar{p}(Z,x) > 0$ for each $x \in Z$.

Definition 2.2 (Kostyrko, P., 2008): Let (Y,ρ) be a metric space and let Z be subset of X. Then, the set Z is said to be porous in X if $\forall\, x \in S(A)$ and $\delta > 0$ there exist some $y \in B(x,\delta)$ and a number $t > 0$ such that $\rho(x,y) + t < \delta$.
or open ball of center 'a' and a radius 'r'.

3.0 Some known results

Theorem 3.1 [Kostyrko, P., (2008)]: Let $A = (a_{nk})$ be a regular matrix, then the convergence field F(A) of the matrix (A) is a δ-porous set in S(A)

Theorem 3.2 [Šalát, T., (1976)]: Let $T = (a_{nk})$ and let (T) be a regular method. Let M_k (k=1,2,…) be a non-void set of real complex such that $Y = M_1 * M_2 * M_3 * \ . \ . \ .$ endowed with the metric ρ. Now, suppose that
(j) Sup diam $M_k < \infty$ and n = 1,2,…
(ii) There exist two sequences $x = \{\xi_k\}_{k=1}^{\infty} \in Y_1(T)$ and $y = \{\eta_k\}_{k=1}^{\infty} \in Y_1(T)$
such that $\{\xi_k - \eta_k\}_{k=1}^{\infty}$ is a convergence sequences and $\{\xi_k - \eta_k\}_{k=1}^{\infty} \neq 0$. Then the set $Y_1(T)$ is a dense set in Y of the Baire first category

Theorem 3.3 [Letavaj, P., (2012)]: Let (S,ρ) be the space of all bounded sequence of real or complex numbers endowed with the metric 'ρ'. Then, the convergence field F(A), is a very porous set in S.

Theorem 3.4 [Siddiqui and Nkuno, (2020)]: Let $A = (a_{nk})$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, the convergence field F(A) is porous in S(A) if and only if the matrix $A = (a_{nk})$ is irregular.

Theorem 3.5 Siddiqui and Nkuno, (2020)]: Let $A = (a_{nk})$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, the convergence field F(A) is close in S(A) if and only if F(A) is porous in S(A).

Theorem 3.6 [Siddiqui and Nkuno, (2020)]: If $A = (a_{nk})$ is any infinite matrix and S is the space of all sequences of real or complex numbers. Then, the convergence field F(A) is dense in S(A) if and only if F(A) is not porous in S(A).

4. Main results

Theorem 4.1 (The necessary and sufficient conditions for F(A) to be δ-porous in S(A)): The space F(A) is δ-porous in S(A) if and only if for any given matrix $A = (a_{nk})$, there exist a sequence $\{A_i\}_{i=1}^{\infty}$ of subset of the matrix $A = (a_{nk})$ where $A_i = (a^i_{nk})$ such that:

i. $A = \cup_{i=1}^{\infty} A_i$

ii. $\cap_{i=1}^{\infty} A_i = \emptyset$.

Proof:

i. Suppose first that the space F(A) is δ-porous in S(A), then, it will follow that F(A) can be expressed as a countable union of porous sets in S(A). Hence,
$F(A) = \cup_{i=1}^{\infty} G_i$
Where $\{G_i\}_{i=1}^{\infty}$ is a sequence of some porous subsets of S(A). Now, since each G_i is porous in S(A) for each i, it will follows that:
$P(x,G_i) > 0$
But then,

$$P(x,G_i) = \lim_{\delta \to 0^+} \text{Sup}\frac{\lambda(x, \delta, G_i)}{\delta}$$

$$\Rightarrow \lim_{\delta \to 0^+} \text{Sup}\frac{\lambda(x, \delta, G_i)}{\delta} > 0$$

$$\Rightarrow \lambda(x, \delta, G_i) > 0$$

But also,

$\lambda(x, \delta, G_i) = \text{Sup}\{t_i : \exists\, y \in B(x,\delta_i) : B(y,t_i) \subset B(x,\delta_i) \text{ and } B(y,t_i) \cap G_i = \emptyset\}$

$\Rightarrow \text{Sup}\{t_i : \exists\, y \in B(x,\delta_i) : B(y,t_i) \subset B(x,\delta_i) \text{ and } B(y,t_i) \cap G_i = \emptyset\} > 0$

This implies that, there exist a sequence $\{t_i > 0\}_{i=1}^{\infty}$ of some numbers such that

$F(A) = \cup_{i=1}^{\infty} G_i$

Thus, since $\{t_i > 0\}_{i=1}^{\infty}$ exist for each i, it follows that there also exist a sequence $\{A_i\}_{i=1}^{\infty}$ of subset of the matrix $A = (a_{nk})$ such that the set $\{G_i\}_{i=1}^{\infty}$ is porous in $S(A)$. Moreover, since the set $\{G_i\}_{i=1}^{\infty}$ is porous in $S(A)$ for each i, $A_i \subset A$. But then,

$F(A) = \cup_{i=1}^{\infty} G_i$ and that $A_i \subset A$. Hence, $= \cup_{i=1}^{\infty} A_i$.

Conversely, suppose that there exist some sequence $\{A_i\}_{i=1}^{\infty}$ of subset of the matrix $A = (a_{nk})$ such that:

$\cup_{i=1}^{\infty} A_i = A$, then it will follow that $\cap_{i=1}^{\infty} F(A_i) = \emptyset$ which implies that each $F(A_i)$ is porous in $S(A)$. But then, $\cup_{i=1}^{\infty} A_i = A$ which also implies that

$F(A) = \cup_{i=1}^{\infty} F(A_i)$ for each i.

Thus, $F(A)$ is a union of some porous of in $S(A)$. Furthermore, since

$F(A) = \cup_{i=1}^{\infty} F(A_i)$ for each i and that $i \in N$, it follows that $\cup_{i=1}^{\infty} F(A_i)$ is countable and therefore $F(A) = \cup_{i=1}^{\infty} F(A_i)$ is a countable union of porous set in $S(A)$ and this ends the proof of the first part of the theorem.

ii. Suppose that the space $F(A)$ is δ-porous in $S(A)$, then, it will follow from part i that $F(A) = \cup_{i=1}^{\infty} F(A_i)$ where each $F(A_i)$ is a porous set in $S(A)$. Now, since $F(A_i)$ is porous in $S(A)$ for each i, it follows from i above that there exist a sequence there exist a sequence $\{t_i > 0\}_{i=1}^{\infty}$ of some numbers such that $\cap_{i=1}^{\infty} F(A_i) = \emptyset$ this implies that there exist the sequence $\{A_i\}_{i=1}^{\infty}$ of subset of the matrix $A = (a_{nk})$ that $\cap_{i=1}^{\infty} A_i = \emptyset$.

Conversely, Suppose that $\cap_{i=1}^{\infty} A_i = \emptyset$ for each i, then, it will follow that

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