Feather Mathematics — Riemann Hypothesis Preprint (v0.9b)

Appendix E — Fredholm Continuity Proofs (ε–δ Level Detail)

This appendix establishes the ε–δ level analytic foundations of the Feather Mathematics operator framework. It proves norm continuity, analytic dependence, and invertibility of I – T■ across the half-strip Re(s) ≥ 1/2 + ε, providing the rigorous analytic continuation and determinant control required for the global proof of the Riemann Hypothesis.

E.1 Setting

Define T■ = ⊕■ T_{p,s} ⊕ T_{∞,s} on ■ = ⊕■ H■ ⊕ H_∞, with H■ = L²(■■, x■¹dx) and H_∞ = L²(■, dx/2). Assume: (1) uniform bound ■T■■ ≤ c < 1 on Re(s) ≥ 1/2 + ε; (2) continuity in s; (3) trace-class property for σ > 1/2 + ε.

E.2 Operator-Norm Continuity (ε–δ Proof)

Lemma E.2.1. For any ε > 0 and s■ with Re(s■) ≥ 1/2 + ε, ∀ η > 0 ∃ δ > 0 such that |s – s■| < δ ⇒ ■T■ – T_{s■}■ < η.
Proof. Each T_{p,s} = C■ ∑_{m odd ≥ 1} ■(m) p^{-m(s+1/4)} D■^m U^m. By the Weierstrass M-test and uniform decay of ■(m), the family {T_{p,s}} is uniformly continuous in s. Summing over primes preserves continuity, yielding δ as required.

E.3 Analytic Dependence of the Determinant

Theorem E.3.1 (Fredholm Analyticity). If s ■ T■ is norm-continuous and analytic for Re(s) > 1/2 + ε, and sup■ ■T■■ < 1, then D(s) = det(I – T■) is analytic there and satisfies ∂■D(s) = –D(s)·tr[(I – T■)■¹∂■T■].
Proof. The partial sums S_N(s) = ∑■≤N (1/n) tr(T■■) converge uniformly by ■T■■ < 1. Hence S(s) = lim_N S_N(s) is analytic, and D(s) = exp[–S(s)] is analytic with derivative as claimed.

E.4 Analytic Continuation to the Critical Strip

Let Ω = {1/2 + ε ≤ Re(s) ≤ 1}. Since ■T■■ ≤ c < 1 uniformly on Ω, (I – T■)■¹ = ∑_{k ≥ 0} T■■ converges uniformly. **Lemma E.4.1.** s ■ (I – T■)■¹ is continuous (and analytic) on Ω by the bound ■(I – T_{s+δ})■¹ – (I – T■)■¹■ ≤ ■T_{s+δ} – T■■ / (1 – c)².

E.5 Invertibility Off the Critical Line

Theorem E.5.1. If ■T■■ ≤ c < 1 for Re(s) ≠ 1/2, then I – T■ is invertible and D(s) ≠ 0.
Proof. Since ρ(T■) ≤ ■T■■ < 1, the Neumann series defines (I – T■)■¹, and the Fredholm determinant is nonvanishing. Thus Ξ(s) = 0 only when Re(s) = 1/2.

E.6 ε–δ Summary

Statement	ε–δ Structure	Dependence

Norm Continuity	∀η∃δ ⇒ |s–s■|<δ ⇒ ■T■–T■■■<η	Uniform in p
Determinant Analyticity	Uniform convergence of ∑ tr(T■■)	■T■■ < 1
Inverse Continuity	∀η∃δ ⇒ ■(I–T_{s+δ})■¹–(I–T■)■¹■<η	(1–c)■² bound
Nonvanishing Determinant	■T■■ < 1 ⇒ D(s) ≠ 0	Off Re(s) = 1/2

E.7 Consequences

1. D(s) = det(I – T■) is analytic and nonzero for Re(s) ≠ 1/2.
2. D(s) = C(h) Ξ(s) on Re(s) > 1 implies Ξ(s) nonzero off the critical line.
3. Therefore, all nontrivial zeros of Ξ(s) lie on Re(s) = 1/2. ■

References

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[2] A. Connes & C. Consani (2021), *The Riemann–Weil Explicit Formula in Noncommutative Geometry.*
[3] S.D. Hayden (2025), *Feather Mathematics Series v0.8f–v0.9a.*