ON THE METRIC AND THE TOPOLOGICAL PROPERTIES OF THE CONVERGENCE FIELD OF REGULAR MATRIX TRANSFORMATIONS

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ABSTRACT: *It has been shown by some researchers that the convergence field* $F(A)$ *of regular matrix transformation is a very porous set in the space* S *of all sequence of real or complex numbers while in [11], it have been proven to be σ-porous set in the linear metric space* $S(A)$ *endowed with Fréchet metric. Also, the usefulness of the well-known theorem on discontinuity points of function of the first bare class has been presented. Thus in this paper, we proved that the convergence field of a various matrix transformation* $F(A)$ *is a metric space as well as a topological space. Further, we have shown that the space is normal and first countable; compact and complete and have unique fix point.*

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1.0 Introduction

It was observed that when a (divergent) sequence, say $\{x_k\}$ is acted upon linearly by an infinite matrix say, $A = (a_{nk})$, then the resultant sequence $\{t_n\}$, given by

$$t_n = \sum_{k=1}^{\infty} a_{nk} x_k \qquad (1.1)$$

may exist and converge. The set of all such sequences obtained (which exist and converge) using this infinite matrix is called the convergence domain of the given matrix.

Thus, if S denotes the set of all the sequences and $S(A)$ is the set of all those sequences for which the sequence given in (1.1) exists, then the convergence domain of the matrix $A = (a_{nk})$ is given by

$$F_A = \left\{x \in S(A) \mid \text{there exist } \lim_{n\to\infty} t_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} x_k \right\} \qquad (1.2)$$

The necessary and sufficient conditions under which the matrix, $A = (a_{nk})$ transform a divergent sequences into convergent sequence (or in other words that A is a regular matrix).

Theorem 1.1(Gordon M. Petersen, 1966): The matrix $A = (a_{nk})$ is regular if the following conditions are satisfied:

i. $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for each $n = 1, 2, \ldots$

ii. $\lim_{n\to\infty} a_{nk} = 0$

iii. $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$

The convergence field of regular matrix transformation have been studied by some researchers several ago. For instance, Mazur et al (1933) found that if the bounded convergence field of a regular matrix $A = (a_{nk})$ is contained in that of another regular matrix $B = (b_{nk})$, then the matrices are consistent on the first field i.e. $A\text{-}\lim x = B\text{-}\lim x$ while in the year 1976, Šalát showed the usefulness of the theorem of the discontinuity points of functions of the first Baire category in the study of the convergence field of a regular matrix method. Kostyrko (2004) discussed some properties of the convergence field of regular matrix transformation of bounded sequences of real numbers and T. Visnyai (2006) proved a generalization of Steinhaus theorem for sequences of a branch space and showed that the result of Salat (1976) and Kostyrko (2004) can be generalized for a space of sequences of element of a Banach space $(X, \|.\|)$. Later Kostyrko (2008) showed that the convergence field $F(A)$ of a regular matrix transformation is a δ-porous set in the metric space $S(A)$ endowed with Fréchet metric. In his own view, Peter Letavaj (2012) showed that the convergence field $F(A)$ of a regular matrix transformation is not only δ-porous but a very porous set in S.

Thus in this paper, we extended the concept of metric space and topological space to the convergence field F(A) of regular matrix transformation using the concept of porosity discussed so far other researchers. We have shown that the space is a compact and complete metric space with unique fix point. Further, we showed that the space F(A) is a linear metric topological sub-space S(A).

2.0 Preliminaries

Definition 2.1 (Šalát, 1976): Let (Y,ρ) be a metric space and let Z be subset of X. Define $\lambda(Z,x,r) = \sup\{t>0 : \exists\, z \in B(x,r); B(z,t) \subset B(x,r) \text{ and } B(z,t) \cap Z = \varnothing\}$.

The upper porosity of the set Z at the point x and the upper porosity of Z are defined as
$$\overline{p}(Z,x) = \lim_{r \to 0} \sup \frac{\lambda(Z,\ x,\ r)}{r} \quad \text{and} \quad \overline{p}(Z) = \inf\{\overline{p}(Z,x) : x \in Z\}.$$

We say that Z is porous if $\overline{p}(Z,x) > 0$ for each $x \in Z$.

Definition 2.2 (Kostyrko, P., 2008): Let (Y,ρ) be a metric space and let Z be subset of X. Then, the set Z is said to be porous in X if $\forall\, x \in S(A)$ and $\delta > 0$ there exist some $y \in B(x,\delta)$ and a number $t > 0$ such that $\rho(x,y) + t < \delta$.

Definition 2.3 (Morris, Sydney A., 2019): Let $a \in X$, where (X,d) is a metric space. Then, for $r > 0$, $S(a,x) = \{x \in X / d(a,x) < r\}$ is a neighborhood or open sphere or open ball of center 'a' and a radius 'r'.

3.0 Some known results

Theorem 3.1 [Kostyrko, P., (2008)]: Let $A = \left(a_{nk}\right)$ be a regular matrix, then the convergence field F(A) of the matrix (A) is a δ-porous set in S(A)

Theorem 3.2 [Šalát, T., (1976)]: Let $T = \left(a_{nk}\right)$ and let (T) be a regular method. Let M_k (k=1,2,…) be a non-void set of real complex such that $Y = M_1 * M_2 * M_3 * \ . \ . \ .$ endowed with the metric ρ. Now, suppose that

(j) Sup diam $M_k < \infty$ and $n = 1,2,\ldots$

(ii) There exist two sequences $x = \{\xi_k\}_{k=1}^{\infty} \in Y_1(T)$ and $y = \{\eta_k\}_{k=1}^{\infty} \in Y_1(T)$ such that $\{\xi_k - \eta_k\}_{k=1}^{\infty}$ is a convergence sequences and $\{\xi_k - \eta_k\}_{k=1}^{\infty} \neq 0$. Then the set $Y_1(T)$ is a dense set in Y of the Baire first category

Theorem 3.3 [Letavaj, P., (2012)]: Let (S,ρ) be the space of all bounded sequence of real or complex numbers endowed with the metric 'ρ'. Then, the convergence field F(A), is a very porous set in S.

Theorem 3.4 [Siddiqui and Nkuno, (2020)]: Let $A = \left(a_{nk}\right)$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, the convergence field F(A) is porous in S(A) if and only if the matrix $A = \left(a_{nk}\right)$ is irregular.

Theorem 3.5 Siddiqui and Nkuno, (2020)]: Let $A = \left(a_{nk}\right)$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, the convergence field F(A) is close in S(A) if and only if F(A) is porous in S(A).

Theorem 3.6 [Siddiqui and Nkuno, (2020)]: If $A = \left(a_{nk}\right)$ is any infinite matrix and S is the space of all sequences of real or complex numbers. Then, the convergence field F(A) is dense in S(A) if and only if F(A) is not porous in S(A).

4. Main results

4.1.0 Metric

[redacted]
depends on the conditions imposed on the matrix $A = \left(a_{nk}\right)$, it follows from Theorem 3.5 that [redacted]
[redacted] irregular leading to Definition 2.13. But then, if such a number $t > 0$ does not exist such that $\rho(x,y) + t < \delta$, then $\rho(x,y) < \delta$ and theorem 3.7 follows.

Thus we define a metric ρ on the space [redacted]

[redacted] and $y = \{y_n\}_{k=1}^{\infty}$.

[redacted]

Further, ∀ x,y ∈ F(A) where x = { [redacted] $\}_{k=1}^{\infty}$ [redacted] y = { [redacted] $\}_{k=1}^{\infty}$ [redacted]

[redacted]

$\Longrightarrow$ [redacted]

$\Leftrightarrow$ [redacted]

$\Leftrightarrow$ [redacted]

$\Leftrightarrow$ x = y

Again

[redacted]

Finally, let ,z,y ∈ F(A) where x = { [redacted] $\}_{k=1}^{\infty}$, [redacted] { [redacted] $\}_{k=1}^{\infty}$ [redacted] y = { [redacted] $\}_{k=1}^{\infty}$ [redacted]

[redacted]

Thus [redacted] hence, the space F(A) together with the metric 'ρ' defined above is a metric space.

Theorem 4.1.1: [redacted] matrix, A = $(a_{n,k})$ [redacted] ([redacted] $\}_{k=1}^{\infty}$ [redacted] { [redacted] $\}_{k=1}^{\infty}$ [redacted](A).

Proof

Let $\{x_m\}$ be any fundamental sequence in F(A) where $x_m = \{x_k^m\}_{k=1}^{\infty}$, and $x_n = \{x_k^n\}_{k=1}^{\infty}$

Then, for each ε > 0, there exist a number N such that

[redacted]

Clearly, $\{x_m\}$ is a fundamental sequence in [redacted]

[redacted] fundamental, it converges. Thus, let $x_m \to x$ as $m \to \infty$ [redacted] x_2^m, x_3^m, ...) and $x = (x_1, x_2, x_3, \dots)$.

Further, we observe that by keeping k fixed, $x_k^m \to x_k$ as $m \to \infty$.

This implies that $\{x_m\}$ converges to x_k componentwise.

But then, we observed from theorem 3.5 that F(A) is a closed subset of S(A). Therefore, it contains all of its points and thus contains all its limit point. Hence, x_k belong to F(A) which in turn implies that x belong to F(A) since $x = (x_k)_{k=1}^{\infty}$.

Thus, if we take ε > 0 for some number N, then, it will follow that

[redacted]

$\Longrightarrow x_k^m \to x_k$

$\Longleftrightarrow(x_1^m ,\ x_2^m,\ x_3^m, \ldots)\longrightarrow(x_1,\ x_2,\ x_3, \ldots)$,

This implies that the sequence $\{x_m\}$ converges in [redacted]

[redacted]

Clearly, since [redacted](A), it follows that

‖x‖ ≥ 0 [redacted](A).

Also

‖x‖ = 0

⟹ [redacted]

⟹ [redacted]

⇔ [redacted]

⇔ [redacted]

$\Longrightarrow x_n = 0$ for each n ∈ N

$\Longrightarrow\{x_1= x_2=x_3= \ . \ . \ .=0\}$

⟹x = 0

Further, for any scalar λ ∈ R, we have that

‖λx‖ = [redacted]

[redacted]

⟹‖λx‖ [redacted]

Finally, let $x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty}$ ∈ F(A). Then,

‖x+y‖ = [redacted]

[redacted]

⟹‖x+y‖ ≤ [redacted]

[redacted]

Theorem 4.1.4: [redacted]pact if and only if it is totally bounded and complete.

Proof:

Proving the compactness of F(A) implies proving its completeness which has been dealt with in theorem 4.2 above. Therefore, this together with the fact that every countably compact metric space is totally bounded shows that F(A) is totally bounded and complete.

Conversely, suppose that F(A) is totally bounded and complete, and $\{x_m\}_{=1}^{\infty}$ where $x_m = \{x_k^m\}_{k=1}^{\infty}$ is any infinite sequence of distinct sequences in F(A). Let N_1 be a finite 1-net for F(A). Now, construct an infinitely many closed sphere of radius 1 about every sequences of N_1 such that they cover F(A). Then, at least one of these spheres, say S_1, will contains an infinite subsequence where $x_i^1 = \{x_{k_i}^1\}_{i=1}^{\infty}$ of the sequence $\{x_m\}_{m=}^{\infty}1$. In the same way, let N_2 be a finite $\frac{1}{2}$-net of $F(A)$ and construct a closed sphere of radius $\frac{1}{2}$ for every point of N_2. Then, at least one of these spheres, say S_2 will contains an infinite subsequence $\{x_i^2\}_{i=1}^{\infty}$ where $x_i^2 = \{x_{k_i}^2\}_{i=1}^{\infty}$ of the sequence $\{x_i^1\}_{i=1}^{\infty}$. By continuing this construction indefinitely in this process, we obtain a nested sequence of closed spheres $S_1 \supset S_2 \supset \ . \ . \ .$ where S_m has the radius $r_m = \frac{1}{2^{m-1}}$.

If suppose that $S_m^{'}$ is a closed sphere with the same center as S_n but with a radius $r_n^{'}$, twice as large as the radius of S_m (i.e $r_m^{'} = \frac{1}{2^{m-2}}$), then clearly, $S_1^{'} \supset S_2^{'} \supset . . . S_m^{'} \supset \ . \ . \ . \ .$ and moreover, $r_m \rightarrow 0$ as $m \rightarrow \infty$. Thus by the nested sphere theorem, it follows that $\cap_{m=1}^{\infty} S_m^{'} \neq \varnothing$; In fact, there is a sequence $\{x_k^o\}_{k=1}^{\infty} = x_o \in F(A)$ such that $\cap_{m=1}^{\infty} S_m^{'} = \{x_k^o\}_{k=1}^{\infty} = x_o$. Clearly, $\{x_k^o\}_{k=1}^{\infty} = x_o$ is a limit point of the original sequence $\{x_m\}_{m=1}^{\infty}$ since every neighborhood of $\alpha_o = \{x_k^o\}_{k=1}^{\infty}$ contains some sphere S_j and hence some infinite sub-sequence$\{x_k^j\}_{k=1}^{\infty}$ where $x_k^j = \{x_{k_i}^j\}_{i=1}^{\infty}$. Therefore, every infinite sequence $\{x_m\}_{m=1}^{\infty}$ of distinct sequences of $F(A)$ has limit point in $F(A)$. Therefore, $F(A)$ is countably compact and hence compact since every countably compact metric space is compact.

4.2.0 TOPOLOGISATION OF F(A)

In transiting from metric space to topological space, we shall collect all the open subset $O_{S(A)}$ and $O_{F(A)}$ of $S(A)$ and $F(A)$ respectively such that each open subset of $F(A)$ is the intersection of the open subset of S(A) with F(A).

Further, we have seen that if the matrix $A = (a_{nk})$ is irregular, F(A) becomes porous in S(A) by theorem 3.5 and if the matrix $A = (a_{nk})$ is regular, F(A) becomes dense in S(A) by theorem 3.7. Thus, we shall define a topology in F(A) as follows:

Lemma 4.2.1: Let $O_{S(A)}$ and $O_{F(A)}$ be the collections of all the open subsets of S(A) and F(A) with respect to the metrics d and $\rho = d|F(A)$ respectively such that each open subset of F(A) is the intersection of an open subset of S(A) with F(A). [redacted] matrix, $A = (a_{n,k})$ [redacted] fined by

$T_{F(A)} = \{U \cap F(A) \setminus U \in T\}$.

Proof:

Let $O_{S(A)}$ and $O_{F(A)}$ be the collections of all the open subsets of S(A) and F(A) with respect to the metrics d and $\rho = d|F(A)$ respectively such that each open subset of F(A) is the intersection of an open subset of S(A) with F(A). Let $U \in O_{F(A)}$. Then, there exist $W \in O_{S(A)}$ such that $U = W \cap F(A)$.

Thus if $T_{F(A)} = \{U \cap F(A) \setminus U \in T\}$ is a topology on F(A), then obviously,

i. $\emptyset = \emptyset \cap F(A) \in T_{F(A)}$ and $F(A) = S(A) \cap F(A) \in T_{F(A)}$.

ii. Further, let $\{U_\alpha\}_{\alpha \in \Lambda} \subseteq T_{F(A)}$. Then, by the definition of $T_{F(A)}$, we have that for each $\alpha \in \Lambda$, there exists some $W_\alpha \in O_{S(A)}$ such that $U_\alpha = F(A) \cap W_\alpha$.

Again, since $W = \cup_{\alpha \in \Lambda} W_\alpha \in O_{S(A)}$,

$U = \cup_{\alpha \in \Lambda} U_\alpha = \cup_{\alpha \in \Lambda}(W_\alpha \cap F(A)) = (\cup_{\alpha \in \Lambda} W_\alpha) \cap F(A) = W \cap F(A) \in T_{F(A)}$.

iii. Let $\{U_1, U_2, U_3, \ldots U_n\} \subseteq T_{F(A)}$. Thus, for each i there exist some $W_i \in O_{S(A)}$ such that $U_i = W_i \cap F(A)$.

Since $O_{S(A)}$ is a collections of all the open subset of S(A) , we have

$U = \cap_{i=1}^{n} U_i = \cap_{i=1}^{n}(W_i \cap F(A)) = (\cap_{i=1}^{n} W_i) \cap F(A) \in T_{F(A)}$.

Therefore, $T_{F(A)}$ having satisfied the above three conditions is a topology on F(A). Thus, the set F(A) together with the topology $T_{F(A)}$ is a topological space and shall henceforth be denoted by $(F(A), T_{F(A)})$.

References

1. Aggarwal, J. (2016), “Introduction To Metric Spaces”
2. Chatterjee, D. (2009), “Topology, General and Algebraic”, ISBN (13) : 978-81-224-2704-2
3. Chidume, C. E. (1989), “Functional analysis: An introduction to metric space”, University of Nsukka Printing Press.
4. Clarke, J. M, ( ), “The intersection of topological and Metric spaces”, p. 2
5. Gonzalez, A.(2017), “Metric and topological spaces”, p.16-17, 75-76
6. Heil, C. (2020), “A Brief Guide to Metrics, Norms, and Inner Products”.
7. https://proofwiki.org/wiki/metric_space_is=first_countable
8. Khrennikov, A. (2016), “Lecture Note In Topology”
9. Kelley, J. R. (1955), “General topology”.
10. Kostyrko, P. (2008),*“Convergence fields of regular matrix transformations 2”*,*Tatra Mt.*
11. Letavaj, P. (2012),*“Convergence field of Abel’s summation method”*, Math. Slovaca 62 (2012), No. 3, 525-530,*Math. Pub* l. 40, 143–147.
12. Munkres, J. R. (2018), “Lecture notes on Topology”.
13. Modak, S. (2011), *“Remarks on dense set”* International Mathematical Forum, Vol. 6, 2011, no. 44, 2153 – 2158
14. Morris, Sydney A. (2019), “Topology without tears^2”
15. Prof. Denis Auroux, (2019), “Introduction To Topological^1
16. Šalát,T.(1976), *“On convergence fields of regular matrix transformations”*, Czechoslovak

Mathematical journal, vol.26(1976), No.4,613-627

17. Siddiqui, Z. U.and Nkuno, D. I. (2020), “On convergence field of some regular matrix transformation”. WJIR Volume 8 Issue 5 (May 2020 Volume 8)
18. Vaidyanathan, P. (2017-2018), “General topology”
19. Visnyai, T. (2006), *“Convergence field of regular matrix transformations of sequences of elements of Banach spaces”* , Miskolc Mathematics notes , Vol. 7 (2006), No 1, pp. 101 – 108
20. Wilkin, D. R. (1991-2), “Connected topological space”.
21. ZajÍček, L. (2005), *“On σ-porous sets in abstract spaces”*.