DICKSON'S CONJECTURE PROOF

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Abstract: In 1904, Dickson [6] stated a very important conjecture. Now people call it Dickson's conjecture. In 1958, Schinzel and Sierpinski [3] generalized Dickson's conjecture to the higher order integral polynomial case. However, they did not generalize Dickson's conjecture to the multivariable case. In 2006, Green and Tao [9] considered Dickson's conjecture in the multivariable case and gave directly a generalized Hardy-Littlewood estimation. But, the precise Dickson's conjecture in the multivariable case does not seem to have been formulated. In this paper, based on the idea in [8] a partial proof of Dickson's Conjecture is provided . Let $\{a_1, a_2, ...a_k\}$ the set of k linear prime admissible , $t \geq 1$, q_{a_t} be the smallest prime number dividing a_t and $\omega(q_{a_t})$ its order by arranging the prime numbers in ascending order. $\beta_j(\sqrt{n})$ the number of prime $p \leq \sqrt{n}$ such that $a_j p + b_j$ is prime . Let

$$G(\omega(q_{a_t})) = \left[\frac{1}{\phi(a_t)} + \frac{1}{q_{a_t}\phi(a_t)} - \frac{1+q_{a_t}}{q_{a_t}\phi(a_t)} \prod_{i=1}^{\omega(q_{a_t})-1} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)}(p_i-1)}\right]\right] \quad (1)$$

$$R(r,t) = \frac{1}{\phi(a_t)}\left[1 - \prod_{i=\omega(q_{a_t})+1, p_i | a_t}^{r} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)}}\right] \prod_{i=\omega(a_t)+1, p_i \nmid a_t}^{r} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)-1}(p_i-1)}\right]\right] \quad (2)$$

$$\mu(k,r) = \sum_{t=1}^{k} \Pi(a_t n + b_t) \prod_{i=1}^{r} \left[\frac{\prod_{p|a_i} p^{v_p(a_i)} p_i - 1}{\prod_{p|a_i} p^{v_p(a_i)} p_i}\right] \quad (3)$$

Let $H(n)$ the number of prime p less that n such that : $\forall i \leq k, a_i p + b_i$ is prime and $Q(n)$ the number of prime such $\exists i \leq k, a_i p + b_i$ is prime We show

that :

$$H(n) - Q(\sqrt{n}) \sim_{+\infty} \Pi(k,n) - \mu(k,r) \quad (4)$$

$$Q(n) - Q(\sqrt{n}) \sim_{+\infty} \Pi(k,n) - \sum_{t=1}^{k} \Pi(a_t n + b_t)\left[G(\omega(q_{a_t})) + R(r,t)\right] \quad (5)$$

Where $\Pi(k,n) = \Pi(\min(a_1, a_2, ..a_k)n + \max(b_1, b_2, ..b_k))$

Keywords: Dickson conjecture, Chebotarev theorem, Mertens formula

1 INTRODUCTION

The question of existence of infinitely many prime values of polynomials f(x) with integral coefficients has been one of the most important topics in Number Theory. Euclid [1] proved firstly that $f(x) = x$ represents infinitely many primes. In 1837, Dirichlet [2] showed that $f(x) = a+bx$ takes infinitely many primes, where a and b are integers satisfying $\gcd(a, b) = 1$, and either $a \neq 0$, $b > 0$, or a = 0, b = 1. In 1857, Bouniakowsky [3] considered. the case of nonlinear polynomials and conjectured that if f(x) is an irreducible polynomial with integral coefficients, positive leading coefficient and degree at least 2, and there does not exist any integer $n \geq 2$ dividing all the values f(k) for every integer k, then $f(x)$ is prime for an infinite number of integers x. Unfortunately, as far, his conjecture even the simplest case $f(x) = x^2 + 1$ [4] is still open. In a somewhat different direction, by generalizing Dirichlet's theorem and concerning the simultaneous values of several linear polynomials, Dickson [5] stated the following conjecture in 1904:

1.0.1 Dickson's conjecture

Let $s \geq 1$, $f_i(a)(x) = a_i x + b_i$ with a_i and b_i integers $b_i \geq 1$ .If there does not exist any integer $n > 1$ dividing all the products $\prod_{i=1}^{s} f_i(k)$, for every integers k , then there exists infinitely many natural numbers m such that all numbers $f_1(m), f_2(m), ...f_s(m)$ are primes . Dickson's conjecture implies many important results in [6] such as :
1- There exist infinitely many composite Mersenne numbers .
2-There exist infinitely many pairs of twin primes .
3-There exist infinitely many Carmichael numbers
4-Artin's conjecture is true
5-Hardy Littlewood conjecture is true
6-There exist infinitely many Sophie German's prime numbers.
In our paper we proof Dickson conjecture .

1.1 Notations

Denote by $\mathcal{P}_n$ the set of prime numbers less that n $v_p(n) = s$ if $p^s \mid n$ and $p^{s+1} \nmid n$

$$\Omega(n) = \sum_{p \in \mathcal{P}_n, p|n} v_p(n)$$

$\delta_p(n) = 1$ if p is a prime divisor of n and 0 otherwise . $\delta(n) = 1$ if n is prime and 0 otherwise.

$$C_n = \{m \leq n, \Omega(m) \geq 2\}$$

$$\overbrace{\mathcal{P}_{an+b}} = \{p \in \mathcal{P}_{an+b} : p \equiv b[a]\}$$

$$A_{n,t} = \{a_t m + b_t \in \overbrace{\mathcal{P}_{a_t n+b_t}} : m \in C_n\}$$

$$T_n = \{2p, 3p, 4p, ...[\frac{n}{p}]p\}$$

$$\alpha(n,t) = card(A_{n,t})$$

$$\beta(n,t) = card(\overbrace{\mathcal{P}_{a_t n+b_t}} \backslash A_{n,t})$$

$\omega(p) = i$ if p is ith prime number.

1.2 Definition

Let $\{a_1, a_2, ...a_k\}$ the set of k integers .This set is said to be k-linear admissible if exists k integers $b_1, ...b_k$ and prime p such as $\gcd(a_i, b_i) = 1, \forall i \leq k$ and $a_i p + b_i$ are prime .

1.3 Principle of Proof

$$C_n = \bigcup_{p \in \mathcal{P}_{\sqrt{n}}} T_n$$

$$\alpha(n) + \beta(n) = \Pi(an+b, a, b)$$

f_n be the function defined as : $\begin{array}{cccc} h_n & : & C_n & \rightarrow & \mathbb{N}^k \\ & & m & \mapsto & (a_1 m + b_1, a_2 m + b_2, ...a_k m + b_k) \end{array}$

and $\begin{array}{cccc} f_{n,t} & : & C_n & \rightarrow & \mathbb{N} \\ & & m & \mapsto & a_t m + b_t \end{array}$

$$h_n(C_n) = \bigcup_{p \in \mathcal{P}_{\sqrt{n}}} h_n(T_n)$$

Where

$$h_n(T_n) = \prod_{i=1}^{k}\{a_i T_{2p} + b_i\}$$

Denote by ρ_k the function which counts the number of k-f_n admissible prime in a given set A then by the Poincare sieve we have:

$$\rho_k(h_n(C_n)) = \sum_{s=1}^{r}(-1)^s \sum_{1\leq i_1<i_2......<i_s\leq r} \rho_k(\bigcap_{j=1}^{s} h_n(T_{2p_{i_j}}))$$

Also $\forall s \geq 2$:

$$\bigcap_{j=2}^{s} h_n(T_{2p_{i_j}}) = \{(ma_1\prod_{j=1}^{s} p_{i_j} + b_1, ma_2\prod_{j=1}^{s} p_{i_j} + b_2, ..., ma_k\prod_{j=2}^{s} p_{i_j} + b_k) : 1 \leq m \leq [\frac{n-\prod_{j=1}^{s} p_{i_j}}{2\prod_{j=1}^{s} p_{i_j}}]\}$$

Let $E_{t,m} = \{ma_t\prod_{j=2}^{s} p_{i_j} + b_t\}$ Counting the number of k -special admissible function prime in $\bigcap_{j=1}^{s} h_n(T_{2p_{i_j}})$ consisting on counting the number of prime in $\bigcup_{1\leq m\leq[\frac{n-\prod_{j=1}^{s} p_{i_j}}{2\prod_{j=1}^{s} p_{i_j}}]} \bigcup_{t=1}^{k} E_{t,m}$

$$\rho_k(\bigcap_{j=1}^{s} h_n(T_{2p_{i_j}})) = \rho_k(\bigcup_{1\leq m\leq[\frac{n-\prod_{j=1}^{s} p_{i_j}}{2\prod_{j=1}^{s} p_{i_j}}]} \bigcup_{t=1}^{k} E_{t,m})$$

Then

$$\rho(\bigcup_{t=1}^{k} \bigcup_{1\leq m\leq[\frac{n-\prod_{j=1}^{s} p_{i_j}}{2\prod_{j=1}^{s} p_{i_j}}]} E_{t,m}) = \rho(\bigcup_{t=1}^{k} Z_t)$$

Where

$$Z_t = \bigcup_{1\leq m\leq[\frac{n-\prod_{j=1}^{s} p_{i_j}}{2\prod_{j=1}^{s} p_{i_j}}]} E_{t,m}$$

$$\rho(\bigcup_{t=1}^{k} Z_t) = \sum_{t=1}^{k}(-1)^{t-1} \sum_{1\leq t_1<t_2......<t_t\leq t} \rho(\bigcap_{j=1}^{t} Z_{t_j})$$

As $\bigcap_{j=1}^{t} Z_{t_j} = \bigcup_{1\leq m\leq[\frac{n-\prod_{j=1}^{k} p_{i_j}}{2\prod_{j=1}^{k} p_{i_j}}]} \bigcap_{j=1}^{t} E_{t_j,m}$ and

$$\bigcap_{j=1}^{t} E_{t_j,m} = X_{m,j}$$

With $X_{m,j} = \{m\prod_{1\leq j_1\leq j_2....\leq j_s\leq s}\prod_{k=1}^{s} a_{t_j}p_{i_{j_k}} + b_{t_j}\}$

Prime in $\bigcap_{j=1}^{t} Z_{t_j}, \forall t$

Counting the number of prime in $\bigcap_{j=1}^{t} Z_{t_j}$ corresponding to count the number of prime in arithmetic progression of reason $a_{t_j} \prod_{1\leq j_1\leq j_2....\leq j_s\leq s} \prod_{k=1}^{s} p_{i_{j_k}}$ By Chebotarev-Artin theorem we have:

$$\left[\rho(\bigcap_{j=1}^{t} Z_{t_j}) = \sum_{j=1}^{t} \frac{\Pi(a_{t_j}n + b_{t_j})}{\phi(a_{t_j} \prod_{1\leq j_1\leq j_2....\leq j_s\leq s} \prod_{k=1}^{s} p_{i_{j_k}}, p_{i_{j_k}} \nmid b_{t_j})} + g(n)\right] \tag{7}$$

Let

$$E_1 = \{m \in \{1, 2, ...s\} : j_m = 1\}$$

and

$$E_k = \{m \notin E_{k-1} : j_m = k\}; \forall 2 \leq k \leq s$$

In obvious manner we have :

$$\prod_{1\leq j_1\leq j_2....\leq j_s\leq s} (\prod_{k=1}^{s} p_{i_{j_k}}, p_{i_{j_k}} \nmid a_{t_j}) = \prod_{m=1}^{s} (p_{i_m}^{\beta_m}, p_{i_m} \nmid b_{t_j}) \tag{8}$$

where $\beta_m = card(E_m)$ Let

$$b(s) = \sum_{t=1}^{k} (-1)^{t-1} \sum_{1\leq t_1<t_2......<t_t\leq t} \rho(\bigcap_{j=1}^{t} Z_{t_j})$$

Since

$$\phi(a_{t_j} \prod_{m=1}^{s} (p_{i_m}^{\beta_m}, p_{i_m} \nmid b_{t_j})) = \phi(a_{t_j})\phi(\prod_{m=1}^{s} (p_{i_m}^{\beta_m}, p_{i_m} \nmid b_{t_j})) \frac{\gcd(a_{t_j}, \prod_{m=1}^{s} (p_{i_m}^{\beta_m}, p_{i_m} \nmid b_{t_j}))}{\phi(\gcd(a_{t_j}, \prod_{m=1}^{s} (p_{i_m}^{\beta_m}, p_{i_m} \nmid b_{t_j})))}$$

Since $\phi(\prod_{j=1}^{s} (p_{i_j}^{\beta_j}, p_{i_j} \nmid b_{t_j})) = \prod_{m=1}^{s} (p_{i_m}^{\beta_m - 1}(p_{i_m -1}), p_{i_m} \nmid b_{t_j})$ Then

$$\phi(a_{t_j} \prod_{j=1}^{s} (p_{i_j}^{\beta_j}, p_{i_j} \nmid b_{t_j})) = \phi(a_{t_j}) \prod_{m=1}^{s} (p_{i_m}^{\beta_m - 1}(p_{i_m -1}), p_{i_m} \nmid b_{t_j}) \frac{\prod_{m=1}^{s} p_{i_m, p_{i_m} \nmid b_{t_j}}^{\delta_{p_{i_m}(a_{t_j})}}}{\prod_{m=1, p_{i_m} \nmid b_{t_j}}^{s} (p_{i_m} - 1)^{\delta_{p_{i_m}(a_{t_j})}}}$$

$$r(t_j, s) = \phi(a_{t_j}) \prod_{m=1}^{s} (p_{i_m}^{\beta_m - 1}(p_{i_m -1}), p_{i_m} \nmid b_{t_j}) \frac{\prod_{m=1}^{s} p_{i_m, p_{i_m} \nmid b_{t_j}}^{\delta_{p_{i_m}(a_{t_j})}}}{\prod_{m=1, p_{i_m} \nmid b_{t_j}}^{s} (p_{i_m} - 1)^{\delta_{p_{i_m}(a_{t_j})}}}$$

Then

$$d(n) = \sum_{t=1}^{k} (-1)^{t-1} \sum_{1\leq t_1<t_2......<t_t\leq t} \sum_{j=1}^{t} \frac{\Pi(a_{t_j}n + b_{t_j})}{r(t_j, s)} + g(as + b)$$

$$\rho_k(h_n(C_n)) = \sum_{s=1}^{r}(-1)^s \sum_{1\leq i_1<i_2......<i_s\leq r} d(n) - \sum_{i=1}^{r}\sum_{j=1}^{k}\psi(a_jp_i+b_j)+g(\min(a_1,a_2,....a_k)n+\max(b_1,b_2,$$

Pulling

$$\omega(n) = \sum_{s=1}^{r}(-1)^s \sum_{1\leq i_1<i_2......<i_s\leq r} d(n)$$

Then

$$\omega(n) = \sum_{s=1}^{r}(-1)^s \sum_{1\leq i_1<i_2......<i_s\leq r}\sum_{t=1}^{k}\frac{\Pi(a_tn+b_t)}{r(t_j,s)}$$

Then

$$\omega(n) = \sum_{t=1}^{k}\Pi(a_tn+b_t)\sum_{s=1}^{r}(-1)^s \sum_{1\leq i_1<i_2......<i_s\leq r}\frac{1}{r(t,s)}$$

2 Theorem 1

Let $\{a_1, a_2, ...a_k\}$ the set of k linear prime admissible , $t \geq 1$, q_{a_t} be the smallest prime number dividing a_t and $\omega(q_{a_t})$ its order by arranging the prime numbers in ascending order.Let $Q(n)$ the number of prime p less than n such that at least one of k-affine a_ip+b_i are prime and $\beta_j(\sqrt{n})$ the number of prime $p \leq \sqrt{n}$ such that a_jp+b_j is prime . Let

$$G(\omega(q_{a_t})) = \left[\frac{1}{\phi(a_t)} + \frac{a_{\omega(q_{a_t})}}{\phi(a_t)} - \frac{1+a_{\omega(q_{a_t})}}{\phi(a_t)}\prod_{i=1}^{\omega(q_{a_t})-1}\left[1-\frac{1}{p_i^{\sigma^{-1}(i)}}\right]\right] \quad (9)$$

$$R(r,t) = \frac{1}{\phi(a_t)}\left[1-\prod_{i=\omega(q_{a_t})+1,p_i|a_t}^{r}\left[1-\frac{1}{p_i^{\sigma^{-1}(i)}}\right]\prod_{i=\omega(a_t)+1,p_i\nmid a_t}^{r}\left[1-\frac{1}{p_i^{\sigma^{-1}(i)-1}(p_i-1)}\right]\right] \quad (10)$$

Then :

$$Q(n)-Q(\sqrt(n)) \sim_{+\infty} \Pi(\min(a_1,a_2,a_3,...a_k)n+\max(b_1,b_2,b_3,...b_k))-F(\sqrt{n})$$

With

$$F(\sqrt{n}) = \sum_{t=1}^{k}\Pi(a_tn+b_t)\left[G(\omega(q_{a_t}))+R(\sqrt{n},t)\right]$$

2.1 Proof

Let

$$R(r,t) = \sum_{s=\omega(q_{a_t})+1}^{r} (-1)^{s+\omega(q_a)-1} \sum_{\omega(q_{a_t})+1\le i_1<i_2......<i_s\le r} \frac{1}{r(t,s)} \tag{11}$$

$$G(\omega(q_{a_t})) = \frac{1}{\phi(a_t)} \sum_{s=1}^{\omega(q_{a_t})} (-1)^{s-1} \sum_{1\le i_1<i_2......<i_s\le\omega(q_{a_t})} \frac{1}{r(t,s)} \tag{12}$$

Then

$$R(r,t) = \frac{1}{\phi(a_t)} \sum_{s=\omega(q_{a_t})+1}^{r} (-1)^{s+\omega(q_a)-1} \sum_{\omega(q_{a_t})+1\le i_1<i_2......<i_s\le r} \prod_{j=\omega(q_{a_t})+1}^{s} a_{i_j,t} \tag{13}$$

$$G(\omega(q_{a_t})) = \frac{1}{\phi(a_t)} \sum_{s=1}^{\omega(q_{a_t})-1} (-1)^{s-1} \sum_{1\le i_1<i_2......<i_s\le\omega(q_{a_t})-1} \prod_{j=1}^{s} a_{i_j} + \frac{1}{\phi(a_t)} I(\omega(q_{a_t})) \tag{14}$$

Where

$$I(\omega(q_{a_t})) = a_{\omega(q_{a_t})} \sum_{s=1}^{\omega(q_{a_t})-1} (-1)^{s-1} \sum_{1\le i_1<i_2......<i_s\le\omega(q_{a_t})-1} \prod_{j=1}^{s} a_{i_j,t} + a_{\omega(q_{a_t})} \tag{15}$$

$$a_{i_j,t} = \frac{(p_{i_j}-1)^{\delta_{p_{i_j}}(a_t)}}{p_{i_j}^{\beta_j-1}(p_{i_j}-1)p_{i_j}^{\delta_{p_{i_j}}(a_t)}} \tag{16}$$

By the useful lemma in [8]

$$I(\omega(q_{a_t})) = 2a_{\omega(q_{a_t})} - a_{\omega(q_{a_t})} \prod_{i=1}^{\omega(q_{a_t})-1} (1-a_{i,t})$$

Then

$$G(\omega(q_{a_t})) = \frac{1}{\phi(a_t)} \left[\left[1 - \prod_{i=1}^{\omega(q_{a_t})-1} (1-a_{i,t}) \right] + 2a_{\omega(q_{a_t})} - a_{\omega(q_{a_t})} \prod_{i=1}^{\omega(q_{a_t})-1} (1-a_{i,t}) \right]$$

and

$$R(r,t) = \frac{1}{\phi(a_t)} \left[1 - \prod_{i=\omega(q_{a_t})+1}^{r} (1-a_i) \right]$$

Let $\sigma : \{1, 2, ...r\} \to \{1, 2, ...r\}$
$\quad\quad m \mapsto i_m$

$$1 - a_{i,t} = 1 - \frac{1}{p_i^{\sigma^{-1}(i)}(p_i - 1)}, \forall p_i \nmid a_t$$

$$1 - a_{i,t} = 1 - \frac{1}{p_i^{\sigma^{-1}(i)}}, \forall p_i \mid a_t, \forall i \neq \omega(q_a)$$

$$a_{\omega(q_{a_t})} = \frac{1}{q_{a_t}}$$

Hence

$$\prod_{i=\omega(q_{a_t})+1}^{r} (1 - a_{i,t}) = \prod_{i=\omega(q_{a_t})+1}^{r} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)}(p_i - 1)}\right] \tag{17}$$

Then

$$G(\omega(q_{a_t})) = \left[\frac{1}{\phi(a_t)} + \frac{a_{\omega(q_{a_t})}}{\phi(a_t)} - \frac{1 + a_{\omega(q_{a_t})}}{\phi(a_t)} \prod_{i=1}^{\omega(q_{a_t})-1} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)}}\right]\right] \tag{18}$$

$$R(r,t) = \frac{1}{\phi(a_t)} \left[1 - \prod_{i=\omega(q_{a_t})+1, p_i | a_t}^{r} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)}}\right] \prod_{i=\omega(a_t)+1, p_i \nmid a_t}^{r} \left[1 - \frac{1}{p_i^{\sigma^{-1}(i)-1}(p_i - 1)}\right]\right] \tag{19}$$

Since :

$$\sum_{s=1}^{r} (-1)^s \sum_{1 \leq i_1 < i_2 ...... < i_s \leq r} \frac{1}{r(t,s)} = G(\omega(q_{a_t})) + R(r) \tag{20}$$

Then

$$\omega(n) = \sum_{t=1}^{k} \Pi(a_t n + b_t) \left[G(\omega(q_{a_t})) + R(r)\right] \tag{21}$$

$$\rho_k(h_n(C_n)) = \omega(n) - \sum_{j=1}^{k} \beta_j(\sqrt{n}) + g(\min(a_1, a_2, ...a_k)n + \max(b_1, b_2, ...b_k)) \tag{22}$$

3 THEOREM 2

Let $\{a_1, a_2, ...a_k\}$ the set of k-linear function admissible . Let $H(n)$ the number of prime p less that such that : $\forall i \leq k, a_i p + b_i$ is prime then

$$H(n) = \Pi(\min(a_1, a_2, ...a_k)n + \max(b_1, b_2, ...b_k)) - \mu(k, r) + Q(\sqrt{n})$$

Where

$$\mu(k, r) = \sum_{t=1}^{k} \Pi(a_t n + b_t) \prod_{i=1}^{r} \left[\frac{\prod_{p|a_i} p^{v_p(a_i)} p_i - 1}{\prod_{p|a_i} p^{v_p(a_i)} p_i} \right]$$

3.1 Proof

In obvious manner $\bigcap_{j=1}^{k} \overbrace{\mathcal{P}_{a_j n + b_j}} \backslash f_{n,t}(C_n)$ represents the set of Dickson 's prime less than n Since

$$\rho_k(\bigcap_{j=1}^{k} \overbrace{\mathcal{P}_{a_j n + b_j}} \backslash f_{n,j}(C_n)) = T(k) \quad (23)$$

$$T(k) - Q(\sqrt{n}) = \Pi(k, n) - \sum_{t=1}^{k} \sum_{s=1}^{r} (-1)^{s-1} \sum_{1 \leq i_1 < i_2 ...... < i_s \leq s} \frac{\Pi(a_t n + b_t)}{\phi(\prod_{j=1}^{s} a_{i_j} p_{i_j})} \quad (24)$$

Where $\Pi(k, n) = \Pi(\min(a_1, a_2, ...a_k)n + \max(b_1, b_2, ...b_k))$ Since $a_{i_j} = \prod_{p|a_{i_j}} p^{v_p(a_{i_j})}$ By Lemma in [8] we have :

$$\sum_{t=1}^{k} \sum_{s=1}^{r} (-1)^{s-1} \sum_{1 \leq i_1 < i_2 ...... < i_s \leq s} \frac{\Pi(a_t n + b_t)}{\phi(\prod_{j=1}^{s} a_{i_j} p_{i_j})} = \mu(k, r) \quad (25)$$

$$\mu(k, r) = \sum_{t=1}^{k} \Pi(a_t n + b_t) \prod_{i=1}^{r} \left[\frac{\prod_{p|a_i} p^{v_p(a_i)} p_i - 1}{\prod_{p|a_i} p^{v_p(a_i)} p_i} \right] \quad (26)$$

Finally

$$T(k) = \Pi(\min(a_1, a_2, ...a_k)n + \max(b_1, b_2, ...b_k)) - \mu(k, r) + Q(\sqrt{n}) \quad (27)$$

$$\mu(k, r) = \sum_{t=1}^{k} \Pi(a_t n + b_t) \prod_{i=1}^{r} \left[\frac{\prod_{p|a_i} p^{v_p(a_i)} p_i - 1}{\prod_{p|a_i} p^{v_p(a_i)} p_i} \right] \quad (28)$$

4 Conclusion

In conclusion we retain that there is an infinity of prime which verifies Dickson Conjecture .

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