ON EINSTEIN NULL HYPERSURFACES OF $(LCS)_{n+2}$-SPACE FORMS

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ABSTRACT. We show that ascreen null hypersurfaces of an $(n+2)$-dimensional Lorentzian concircular structure $(LCS)_{n+2}$-manifold admits an induced Ricci tensor. We therefore prove, under some geometric conditions, that an Einstein ascreen null hypersurface is locally a product of null curves and products of spheres.

1. INTRODUCTION

Null hypersurfaces appears in general relativity as models of different types of black hole horizons (see [5, 6] for details) and their theory is quite fundamental to modern mathematical physics. The study of null submanifolds in semi-Riemannian manifolds was introduced by Duggal-Benjancu [5] and later updated by Duggal-Sahin [6]. In the above books, the authors laid a foundation for research on null geometry by constracting their structural equations, among other results. In fact, they introduced a non-degenerate screen distribution to construct a null transversal vector bundle which is non-intersecting to its null tangent bundle and developed local geometry of null curves, hypersurfaces and submanifolds. Other pioneers of the theory include D. N. Kupeli [18]–whose approach is purely intrinsic compared to that of [5, 6]. Since then, many researchers including but not limited to; [1, 3, 4, 8, 9, 11], have researched on null submanifolds and many interesting results have been obtained.

Among the most studied null hypersurfaces are those with an integrable screen distribution. They include the well-known screen conformal ones, among others. It was shown in [7], that all screen integrable null hypersurfaces are locally isometric to $\mathcal{C}_\xi \times M'$, where $\mathcal{C}_\xi$ is a null curve tangent to the normal bundle of the hypersurface and M' is a leaf of its screen distribution. In particular, [5] proves that a null cone of an $(n+2)$-dimensional Lorentzian space $\mathbb{R}^{n+2}_1$ is screen conformal, satisfying the above structure, with $M' \cong \mathbb{S}^n$. Under some geometric conditions on the ambient space, Duggal-Sahin [6] also proves that a screen conformal Einstein

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null hypersurface is locally a triple product $\mathcal{C}_\xi \times M_\alpha \times M_\beta$, where M_α and M_β are some leaves of its screen distribution (see Theorem 2.5.17 of [6] for more details).

In the present paper, we consider an ascreen null hypersurface of a Lorentzian concircular structure space form. Such hypersurfaces admits an induced Ricci tensor (see Theorem 3.3), which we use to define Einstein null hypersurfaces for this class. Consequently, we prove a characterization theorem (Theorem 4.2) which indicates further details on the leaves M_α and M_β in this case. In fact, we show, under some geometric conditions, that these leaves are spheres and Euclidean spaces. For details on the geometry of Lorentzian concircular manifolds, we refer the reader to [15, 16] and many more references cited therein. The paper is arranged as follows; In Section 2, we quote some basic notions needed in the rest of the paper. In Section 3, we prove several characterization results on null hypersurfaces of Lorentzian concircular structure manifolds. Finally, in Section 4, we present our main result (Theorem 4.2).

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be an $(n+2)$-dimensional Lorentzian manifold. Throughout this paper, we denote by $\Gamma(E)$ the module of smooth sections of a vector bundle E over $\overline{M}$. A vector field V defined by $\overline{g}(X, V) = A(X)$, for any $X \in \Gamma(T\overline{M})$, is said to be a *concircular* [15, 16] vector field if, for any $X, Y \in \Gamma(T\overline{M})$, we have $(\overline{\nabla}_X A)Y = \alpha(\overline{g}(X, Y) - \omega(X)A(Y))$, where α is a non-vanishing smooth function and ω is a closed 1-form. Here, $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$ with respect to $\overline{g}$. Suppose that $\overline{M}$ admits a unit *timelike* concircular vector field ζ, called the *characteristic* vector field of the manifold. Then, we have

$$\overline{g}(\zeta, \zeta) = -1. \tag{2.1}$$

Since ζ is a unit concircular vector field, it follows that there exists a non-zero 1-form θ such that for

$$\overline{g}(X, \zeta) = \theta(X), \;\; \forall\, X \in \Gamma(T\overline{M}), \tag{2.2}$$

and the following relation holds

$$(\overline{\nabla}_X \theta)Y = \alpha(\overline{g}(X, Y) + \theta(X)\theta(Y)), \tag{2.3}$$

for all $X, Y \in \Gamma(T\overline{M})$, and α is a non-vanishing smooth function satisfying

$$X\alpha = d\alpha(X) = \rho\theta(X). \tag{2.4}$$

Here, ρ is a smooth function given by

$$\rho = -\zeta\alpha. \tag{2.5}$$

Let us put

$$\overline{\phi}X = (1/\alpha)\overline{\nabla}_X\zeta, \quad \forall\, X \in \Gamma(T\overline{M}). \tag{2.6}$$

Then, by (2.3) and (2.6), we have

$$\overline{\phi}X = X + \theta(X)\zeta, \tag{2.7}$$

which follows that $\overline{\phi}$ is a symmetric $(1,1)$ tensor field called the *structure* tensor field of the manifold. Thus, the Lorentzian manifold $\overline{M}$ together with the unit timelike concircular vector field ζ, its associated 1-form θ and a $(1,1)$ tensor field $\overline{\phi}$ is said to be a *Lorentzian concircular structure* manifold (briefly, an $(LCS)_{n+2}$-manifold) [15, 16]. In particular, if $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [14]. In an $(LCS)_{n+2}$-manifold, the following relations hold for all vector fields $X, Y \in \Gamma(T\overline{M})$:

$$\overline{\phi}^2 X = X+\theta(X)\zeta, \quad \overline{\phi}\zeta = 0, \;\; \theta \circ \overline{\phi} = 0, \quad \theta(\zeta) = -1, \tag{2.8}$$

$$\overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) + \theta(X)\theta(Y), \tag{2.9}$$

$$(\overline{\nabla}_X\overline{\phi})Y = \alpha(\overline{g}(X, Y)\zeta + 2\theta(X)\theta(Y)\zeta + \theta(Y)X), \;\; \overline{\nabla}_X\zeta = \alpha\overline{\phi}X. \tag{2.10}$$

Let $\overline{R}$ denote the curvature tensor of $\overline{M}$. Then, the following relation holds

$$\theta(\overline{R}(X, Y)Z) = (\alpha^2 - \rho)(\overline{g}(Y, Z)\theta(X) - \overline{g}(X, Z)\theta(Y)), \tag{2.11}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. A semi-Riemannian manifold $(\overline{M}, \overline{g})$ of constant sectional curvature c is called a semi-Riemannian space form (see [17, p. 80]) and denoted by $\overline{M}(c)$. The curvature tensor field $\overline{R}$ of $\overline{M}(c)$ is given by

$$\overline{R}(X, Y)Z = c(\overline{g}(Y, Z)X - \overline{g}(X, Z)Y), \;\; \forall\, X, Y, Z \in \Gamma(T\overline{M}). \tag{2.12}$$

Let $\overline{M}(c)$ be a $(LCS)_{n+2}$-manifold of constant curvature c. Then by (2.11) and (3.5), it follows that c satisfies

$$c = \alpha^2 - \rho. \tag{2.13}$$

Suppose M is an $(n+1)$-dimensional smooth manifold and $i : M \longrightarrow \overline{M}$ a smooth mapping such that each point $x \in M$ has an open neighborhood $\mathcal{U}$ for which i restricted to $\mathcal{U}$ is one-to-one and $i^{-1} : i(\mathcal{U}) \longrightarrow M$ are smooth. Then, we say that $i(M)$ is an immersed hypersurface of $\overline{M}$. If this condition globally holds, then $i(M)$ is called an embedded hypersurface of $\overline{M}$, which we assume in this paper. The embedded hypersurface has a natural manifold structure inherited from the manifold structure on $\overline{M}$ via the embedding mapping. At each point $i(x)$ of $i(M)$, the tangent space is naturally identified with an $(n+1)$-dimensional subspace $T_{i(x)}M$ of the tangent space $T_{i(x)}\overline{M}$. The embedding i induces, in general, a symmetric tensor field, say g, on $i(M)$ such that $g(X,Y)|_x = \overline{g}(i_*X, i_*Y)|_{i(x)}$, for all $X, Y \in T_xM$. Here, i_* is the differential map of i defined by $i_* : T_xM \longrightarrow T_{i(x)}\overline{M}$ and $(i_xX)f = X(f \circ i)$, for an arbitrary smooth function f in a neighborhood of $i(x)$ of $i(M)$. Henceforth, we write M and x instead of $i(M)$ and $i(x)$. Due to the causal character of three categories (spacelike, timelike and lightlike) of the vector fields of $\overline{M}$, there are three types of hypersurfaces M, namely, *Riemannian*, *semi-Riemannian* and *null* (or *lightlike*) and g is a non-degenerate or a degenerate symmetric tensor field on M according as M is of the first two types and of the third type, respectively. The geometry of Riemannian or semi-Riemannian hypersurfaces is well-known and has received a considerable attention, for example see [17] and many more references cited therein. In the present paper, we focus on null hypersurfaces using the approach of Duggal-Bejancu [5].

Now let g be degenerate on M. Then, there exists a *nonzero* vector field ξ on M such that $g(\xi, X) = 0$, for all $X \in \Gamma(TM)$. The *radical* or the *null space* [17, p. 53] of T_xM, at each point $x \in M$, is a subspace $\operatorname{Rad} T_xM$ defined by

$$\operatorname{Rad} T_xM = \{\xi \in T_xM : g_x(\xi, X) = 0, \ \ \forall\, X \in T_xM\}, \qquad (2.14)$$

whose dimension is called the *nullity degree* of g and M is called a *null hypersurface* of $\overline{M}$. It follows from (2.14) that $T_xM^{\perp}$ is also null and satisfy

$$\operatorname{Rad} T_xM = T_xM \,\cap\, T_xM^{\perp}.$$

For a hypersurface M $\dim(T_xM^{\perp}) = 1$, implies that $\dim(\operatorname{Rad} T_xM) = 1$ and $\operatorname{Rad} T_xM = T_xM^{\perp}$. We call $\operatorname{Rad} TM$ a radical (null) distribution of M. Thus, for a null hypersurface M, TM and $TM^{\perp}$ have a nontrivial intersection and their sum is not the whole of tangent bundle space $T\overline{M}$. In other words, a vector of $T_x\overline{M}$ cannot be decomposed uniquely into a component tangent to T_xM and a component of $T_xM^{\perp}$. Therefore, the standard text-book definition of the second fundamental form and the Gauss-Weingarten formulas do not work, in the usual way, for the null case. To overcome this difficulty, Duggal-Bejancu [5] introduced an approach to null geometry, which we follow in this paper. The approach consists of fixing, on the null hypersurface, a geometric data formed by a null section and a *screen distribution*. By screen distributionon of M, we mean a complementary bundle of $TM^{\perp}$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M is paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$ [18]. We denote by $S(TM)$ the screen distribution over M. Then we have the decompostion

$$TM = S(TM) \perp TM^{\perp}, \tag{2.15}$$

where $\perp$ denotes the orthogonal direct sum. From [5] or [6], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N, \xi) = 1, \;\; \overline{g}(N, N) = \overline{g}(N, W) = 0, \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.16}$$

It then follows that

$$T\overline{M}|_{M} = S(TM) \perp \{TM^{\perp} \oplus tr(TM)\} = TM \oplus tr(TM), \tag{2.17}$$

where $\oplus$ denote the direct (non-orthogal) sum. We call $tr(TM)$ a (null) transversal vector bundle along M. In fact, from (2.16) and (2.17) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the screen distribution $S(TM)$. A vector field N as in (2.16) is called a null transversal vector field of M. It is then noteworthy that the choice of a nulltransversal vector field N along M determines both the null transversal vector bundle, the screen distribution $S(TM)$ and a unique radical vector field, say ξ, satisfying (2.16). The name screen distribution is justified as follows; in the case M is a null cone of a 4-dimensional Lorentz manifold, the integral curves of vector fields in $TM^{\perp}$ are null (lightlike) rays and fibers of $S(TM)$ can be visualized as screen that are transversal to these rays.

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [5]

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.18}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.19}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{2.20}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \quad A^*_{\xi}\xi = 0, \tag{2.21}$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^{\perp})$ and $N \in \Gamma(tr(TM))$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_{ξ} are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$g(A^*_\xi X, Y) = B(X, Y), \quad g(A_N X, PY) = C(X, PY), \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$. Moreover,

$$\overline{g}(A^*_\xi X, N) = 0 \quad \text{and} \quad \overline{g}(A_N X, N) = 0, \tag{2.23}$$

for all $X \in \Gamma(TM)$. From these relations, we notice that A^*_ξ and A_N are both screen-valued operators. Let $\vartheta = \overline{g}(N, \cdot)$ be a 1-form metrically equivalent to N defined on $\overline{M}$. Take $\eta = i^*\vartheta$ to be its restriction on M, where $i : M \to \overline{M}$ is the inclusion map. Then it is easy to show that

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{2.24}$$

for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection. Denote by R and R^* the curvature tensors of the connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae (2.18)-(2.21), we obtain the following curvature relations (see details in [5, 6]).

$$\begin{aligned}\overline{g}(\overline{R}(X, Y)Z, \xi) =& (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) \\ & - \tau(Y)B(X, Z), \end{aligned} \tag{2.25}$$

$$\begin{aligned}\overline{g}(\overline{R}(X, Y)Z, PW) =& g(R(X, Y)Z, PW) + B(X, Z)C(Y, PW) \\ & - B(Y, Z)C(X, PW), \end{aligned} \tag{2.26}$$

$$\begin{aligned}g(R(X, Y)PZ, PW) =& g(R^*(X, Y)PZ, PW) + C(X, PZ)B(Y, PW) \\ & - C(Y, PZ)B(X, PW), \end{aligned} \tag{2.27}$$

$$\overline{g}(\overline{R}(X, Y)\xi, N) = C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X, Y), \tag{2.28}$$

where

$$(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z), \tag{2.29}$$

$$\text{and} \quad 2d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y]), \tag{2.30}$$

for all $X, Y, Z, W \in \Gamma(TM)|_{\mathcal{U}}$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$.

3. BASIC RESULTS ON ASCREEN NULL HYPERSURFACES OF A $(LCS)_{n+2}$ MANIFOLD

Let $(\overline{M}, \overline{g})$ be a $(LCS)_{n+2}$-manifold. Let (M, g) be a null hypersurface of $\overline{M}$. As $\overline{M}$ is Lorenztian, it is obvious that any screen distribution over M is Riemannian. Next, since ζ is a global vector field of $\overline{M}$, we decompose it as follows $\zeta = W + a\xi + bN$, where a and b are smooth functions given by $a = \theta(N)$ and

$b = \theta(\xi)$, and W a smooth section tangent to $S(TM)$. Since ζ is a unit timelike vector field (see relation (2.1)), we have $g(W, W) + 2ab = -1$. Suppose that a or b vanishes, then it follows that $g(W, W) = -1$. Consequently, W is a unit timelike vector field of $S(TM)$. This is a contradiction as $S(TM)$ is Riemannian. Thus, the following hold.

Theorem 3.1. *There exist no null hypersurface of a $(LCS)_{n+2}$-manifold $(\overline{M}, \overline{g})$ such that ζ is tangent or transversal to M.*

In the theory of null hypersurfaces of Sasakian manifolds, it is possible to select a screen distribution $S(TM)$ containing $\overline{\phi}TM^{\perp}$ and $\overline{\phi}tr(TM)$ as subbundles. See [6, 8, 9] for details. This does not hold for null hypersurfaces of a $(LCS)_{n+2}$-manifold $(\overline{M}, \overline{g})$. More precisely, $\overline{\phi}TM^{\perp} \not\subset S(TM)$ and $\overline{\phi}tr(TM) \not\subset S(TM)$. Moreover, $\overline{\phi}TM^{\perp} \cap TM^{\perp} = \{0\}$ and $\overline{\phi}tr(TM) \cap tr(TM) = \{0\}$ (see [12] for more details).

In view of [8], we will say that a null hypersurface (M, g) of a (LCS)-manifold $(\overline{M}, \overline{g})$ is *ascreen* if the characteristic vector field ζ belongs to $S(TM)^{\perp} (= TM^{\perp} \oplus tr(TM))$. Equivalently, M is ascreen if $W = 0$. Therefore, for an ascreen null hypersurface of $(LCS)_{n+2}$, the structure vector field ζ has the following decomposition;

$$\zeta = a\xi + bN, \quad \text{where} \quad a = \theta(N) \quad \text{and} \quad b = \theta(\xi). \tag{3.1}$$

Here, a and b are nonzero smooth functions. Suppose that M is an ascreen null hypersurface of $\overline{M}$. Differentiating (3.1) and using (2.19) and (2.21), we get

$$\alpha\overline{\phi}X = -aA^*_{\xi}X - bA_N X + (Xa - a\tau(X))\xi + (Xb + b\tau(X))N, \tag{3.2}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (3.2) with N and ξ, in turn, we get

$$Xa - a\tau(X) = \alpha\eta(X) + \alpha a\theta(X), \quad Xb + b\tau(X) = \alpha b\theta(X), \tag{3.3}$$

in which we have used the fact $X = PX + \eta(X)\xi$, for any $X \in \Gamma(TM)$. On the other hand, taking the inner product of (3.2) with PY, where $Y \in \Gamma(TM)$, we have

$$aB(X, PY) + bC(X, PY) = -\alpha g(PX, PY). \tag{3.4}$$

Notice from (3.4) that C is symmetric on $S(TM)$. Setting $X = \xi$ in (3.4) and using (2.21) together with the fact $b \neq 0$, we get

$$C(\xi, PY) = 0, \quad \forall Y \in \Gamma(TM). \tag{3.5}$$

It follows from (3.5) that ξ is an eigenvalue of A_N with respect to ξ. Furthermore, it easy to see from (3.3) that there is no M with a or b constant on M and $\tau = 0$. In view of (3.4), we see that C is symmetric and hence, $S(TM)$ is an integrable distribution on M. By similar arguments as in [7] we have the following.

Theorem 3.2. *Any ascreen null hypersurface (M, g) of a $(LCS)_{n+2}$-manifold $(\overline{M}, \overline{g})$ is locally isometric to $\mathcal{C}_\xi \times M'$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M' is a leaf of $S(TM)$.*

It is well-known that the Ricci tensor of null hypersurface (and generally of null submanifold) is not symmetric. This is because the induced connection is not a metric connection (see 2.24). Let (M, g) be a null hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then the Ricci tensor of the induced connection ∇ is symmetric, if and only if, each 1-form τ induced by $S(TM)$ is closed, i.e., $d\tau = 0$, on $\mathcal{U} \subset M$ [5, p. 99] (also see [6]). Setting $Z = \xi$ in (2.11) we get $\theta(\overline{R}(X, Y)\xi) = \overline{g}(\overline{R}(X, Y)\xi, \zeta) = 0$, for any $X, Y \in \Gamma(TM)$. For an ascreen null hypersurface, this last relation reduces to $\overline{g}(\overline{R}(X, Y)\xi, bN) = 0$. As $b \neq 0$, we get

$$\overline{g}(\overline{R}(X, Y)\xi, N) = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{3.6}$$

Now considering (2.28), (2.22), (2.23) and (3.6), we get

$$\begin{aligned} 2d\tau(X, Y) &= C(Y, A^*_\xi X) - C(X, A^*_\xi Y) \\ &= g(A_N Y, A^*_\xi X) - g(A_N X, A^*_\xi Y). \end{aligned} \tag{3.7}$$

In view of (3.4), we have $A_N X = -(a/b)A^*_\xi X - (\alpha/b)PX$, for any $X \in \Gamma(TM)$. Thus, applying this relation to (3.7), we get

$$2d\tau(X, Y) = \frac{\alpha}{b}(B(Y, X) - B(X, Y)) = 0, \quad X, Y \in \Gamma(TM). \tag{3.8}$$

in which we have used the symmetry of B. Thus, by (3.8) we see that $d\tau = 0$. Hence, the following result hold.

Theorem 3.3. *The Ricci tensor of an ascreen null hypersurface (M, g) of any $(LCS)_{n+2}$-manifold $(\overline{M}, \overline{g})$ is symmetric.*

Let $\tilde{\xi} = \lambda\xi$, then it follows that $\tilde{N} = (1/\lambda)N$. Moreover, $\tilde{B} = B$ and

$$\tau(X) = \tilde{\tau}(X) + X\ln(\lambda), \quad \forall\, X \in \Gamma(TM_{|\mathcal{U}}). \tag{3.9}$$

This shows that B and τ depend on the section ξ on $\mathcal{U}$. By Theorem 3.3 and Poincaré's lemma we obtain $\tau(X) = Xf$, where f is a smooth function on $\mathcal{U}$. Let $\lambda = \exp(f)$ in (3.9), then, $\tilde{\tau} = 0$ on $\mathcal{U}$. Thus, we have

Corollary 3.4. *Let (M, g) be an ascreen null hypersurface of a $(LCS)_{n+2}$-manifold $(\overline{M}, \overline{g})$. There exist a pair $\{\xi, N\}$ on $\mathcal{U}$ such that the corresponding 1-form τ from the Weingarten equation vanishes.*

From now on, we consider an ascreen null hypersurface (M, g) of a $(LCS)_{n+2}$-manifold, with the pair $\{\xi, N\}$ of Corollary 3.4.

Consider the quasi-orthonormal frame $\{\xi, W_i\} : 1 \leq i \leq n$, on M, where $TM^{\perp} = \mathrm{Span}\{\xi\}$ and $S(TM) = \mathrm{Span}\{W_i\}$ and let $\{\xi, N, W_i\}$ be the corresponding frames field on $\overline{M}$. Then, the Ricci tensor, Ric, of M is given by

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(R(X, W_i)Y, W_i) + \overline{g}(R(X, \xi)Y, N), \tag{3.10}$$

for any $X, Y \in \Gamma(TM)$. Applying (2.26) to (3.10), we get

$$\begin{aligned} \mathrm{Ric}(X, Y) = \overline{\mathrm{R}}\mathrm{ic}(X, Y) + B(X, Y)\mathrm{tr}A_N - g(A_N X, A^*_\xi Y) \\ - \overline{g}(R(\xi, Y)X, N), \end{aligned} \tag{3.11}$$

where $\overline{\mathrm{R}}\mathrm{ic}$ denotes the Ricci tensor of $\overline{M}$ and $\mathrm{tr}(\cdot)$ the trace operator on $S(TM)$. As M is an ascreen null hypersurface of $\overline{M}$, relation (3.4) implies that

$$\mathrm{tr}A_N = -\frac{a}{b}\mathrm{tr}A^*_\xi - \frac{\alpha}{b}n, \tag{3.12}$$

and $$g(A_N X, A^*_\xi Y) = -\frac{a}{b} g(A^*_\xi X, A^*_\xi Y) - \frac{\alpha}{b} B(X, Y). \tag{3.13}$$

On the other hand, setting $X = \xi$, $Y = X$ and $Z = Y$ in (2.11), and using the fact that $\overline{g}(R(\xi, X)Y, N) = \overline{g}(\overline{R}(\xi, X)Y, N)$, we get

$$\overline{g}(R(\xi, X)Y, N) = (\alpha^2 - \rho)g(X, Y) - \frac{a}{b}\overline{g}(\overline{R}(\xi, X)Y, \xi). \tag{3.14}$$

Replacing (3.12), (3.13) and (3.14) in (3.11), leads to

Lemma 3.5. *The Ricci tensor,* Ric*, of an ascreen null hypersurface of a $(LCS)_{n+2}$-manifold is given by the relation*

$$\begin{aligned} \mathrm{Ric}(X, Y) = \overline{\mathrm{R}}\mathrm{ic}(X, Y) - (\alpha^2 - \rho)g(X, Y) + \left(\frac{\alpha}{b} - \frac{\alpha}{b}n - \frac{a}{b}\mathrm{tr}A^*_\xi\right) B(X, Y) \\ + \frac{a}{b} g(A^*_\xi X, A^*_\xi Y) + \frac{a}{b}\overline{g}(\overline{R}(\xi, X)Y, \xi), \quad \forall\, X, Y \in \Gamma(TM). \end{aligned}$$

4. Main result

In this section, we prove a characterization theorem (Theorem 4.2) of Einstein ascreen null hypersurfaces of a $(LCS)_{n+2}$-space form $\overline{M}(c)$. Since M admits an induced Ricci tensor (see Theorem 3.3), it therefore meaningful to consider such hypersurfaces whose Ricci tensor is proportional to their induced metrics, i.e. Einstein ascreen null hypersurfaces. More explicitly, we say that an ascreen null hypersurface M of a $(LCS)_{n+2}$-manifold is Einstein if there exist a constant γ such that

$$\operatorname{Ric}(X,Y)=\gamma g(X,Y),\quad \forall\, X,Y\in\Gamma(TM). \tag{4.1}$$

Consider a frame field of eigenvectors $\{\xi, E_1,\dots,E_n\}$ of A^*_ξ such that $\{E_1,\dots,E_n\}$ is an orthonormal frame field of $S(TM)$. Then $A^*_\xi E_i=\kappa_i E_i$, for $1\le i\le n$. As M is Einstein and $\overline{M}$ is a space of constant curvature c, we deduce from (3.5), (2.13), (4.1) and Lemma 3.5 that

$$g(A^*_\xi X, A^*_\xi Y)+\left(\frac{\alpha}{a}-\frac{\alpha}{a}n-\operatorname{tr}A^*_\xi\right)g(A^*_\xi X,Y)+\frac{\gamma-nc}{2a^2}g(X,Y)=0, \tag{4.2}$$

in which we have considered the fact that $2ab+1=0$ as M is ascreen. Setting $X=Y=E_i$ in (4.2), it follows that k_i is a solution of

$$x^2+\left(\frac{\alpha}{a}-\frac{\alpha}{a}n-\operatorname{tr}A^*_\xi\right)x+\frac{\gamma-nc}{2a^2}=0. \tag{4.3}$$

It is obvious that (4.3) has at most two distinct solutions which are real-valued functions on $\mathcal{U}$. Assume there exists $k\in\{1,\dots,n\}$ such that $\kappa_1=\ldots=\kappa_k=\lambda_1$ and $\kappa_{k+1}=\ldots=\kappa_n=\lambda_2$, by renumbering if necessary. Then, (4.3) leads to

$$\lambda_1+\lambda_2=-\frac{\alpha}{a}+\frac{\alpha}{a}n+\operatorname{tr}A^*_\xi,\quad\text{and}\quad \lambda_1\lambda_2=\frac{\gamma-nc}{2a^2}. \tag{4.4}$$

As $\operatorname{tr}A^*_\xi=k\lambda_1+(n-k)\lambda_2$, the first relation of (4.4) gives

$$(k-1)\lambda_1+(n-k-1)\lambda_2=\frac{\alpha}{a}(1-n). \tag{4.5}$$

Let us consider the following distributions P_{λ_1} and P_{λ_2} on M;

$$P_{\lambda_1}=\{X\in\Gamma(TM): A^*_\xi X=\lambda_1 PX\},$$
$$P_{\lambda_2}=\{X\in\Gamma(TM): A^*_\xi X=\lambda_2 PX\}.$$

The following result holds;

Lemma 4.1. *For λ_1 and λ_2 distinct, the following hold;*

1. $P_{\lambda_1}\perp_g P_{\lambda_2}$ *and* $P_{\lambda_1}\perp_B P_{\lambda_2}$,
2. $TM=TM^\perp\perp P_{\lambda_1}\perp P_{\lambda_2}$,
3. P_{λ_1} *and* P_{λ_2} *are completely integrable,*
4. $\operatorname{Im}(A^*_\xi-\lambda_1 P)\subset\Gamma(P_{\lambda_2})$ *and* $\operatorname{Im}(A^*_\xi-\lambda_2 P)\subset\Gamma(P_{\lambda_1})$,
5. $d\lambda_1|_{P_{\lambda_1}}=0$ *and* $d\lambda_2|_{P_{\lambda_2}}=0$, *since* $\tau=0$,
6. *for* $X,Y\in\Gamma(P_{\lambda_1})$ *and* $U,V\in\Gamma(P_{\lambda_2})$, *we have*

$$\nabla_X U\in\Gamma(P_{\lambda_2}),\qquad \nabla_U X\in\Gamma(P_{\lambda_1}), \tag{4.6}$$

$$\textit{and}\quad g(\nabla_Y X,U)=0,\quad g(X,\nabla_V U)=0. \tag{4.7}$$

Proof. A proof uses similar arguments as in Lemmas 2.5.6–2.5.14 of [6, p. 79–82]. □

In view of Lemma 4.1, we have the following main result.

Theorem 4.2. *Let (M, g) be an Einstein ascreen null hypersurface of $(LCS)_{n+2}$-manifold $\overline{M}^{n+2}(c) : n \geq 3$ of constant curvature c, and with distinct screen principal curvatures λ_1 and λ_2 with respect to A^*_ξ. Then M is isoparametric. Furthermore, M is locally a null triple product $\mathcal{C}_\xi \times M^k_{\lambda_1}(c_1) \times M^{n-k}_{\lambda_2}(c_2)$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$, and M_{λ_1}, M_{λ_2} are, respectively, k and $(n-k)$-dimensional integral manifolds of P_{λ_1} and P_{λ_2} of constant curvatures*

$$c_i = \left(\alpha - \frac{\lambda_i}{b}\right)^2 - \rho, \quad i = 1, 2, \tag{4.8}$$

where λ_1 and λ_2 satisfy the relations

$$(k-1)\lambda_1 + (n-k-1)\lambda_2 = 2b\alpha(n-1) \quad \textit{and} \quad \lambda_1\lambda_2 = 2b^2(\gamma - nc). \tag{4.9}$$

Moreover, if $\gamma = c = 0$, then M is locally isometric to

$$\mathcal{C}_\xi \times \mathbb{R}^k \times \mathbb{S}^{n-k}\left(\frac{n-k-1}{2\alpha\sqrt{k(n-1)}}\right), \tag{4.10}$$

$$\textit{or} \quad \mathcal{C}_\xi \times \mathbb{S}^k\left(\frac{k-1}{2\alpha\sqrt{(n-1)(n-k)}}\right) \times \mathbb{R}^{n-k}. \tag{4.11}$$

If $\alpha = 1$ and $\gamma = 0$, then M is locally isometric to

$$\mathcal{C}_\xi \times \mathbb{S}^k\left(\frac{b}{b-\sigma}\right) \times \mathbb{S}^{n-k}\left(\frac{\sigma}{\sigma - 2nb}\right), \tag{4.12}$$

where σ is a real solution of

$$\sigma^2 - 2\frac{b(n-1)}{(k-1)}\sigma - \frac{2nb^2(n-k-1)}{(k-1)} = 0. \tag{4.13}$$

Proof. Note from Lemma 4.1 that P_{λ_1} and P_{λ_2} are parallel distributions with respect to ∇^*. More precisely, (4.7) and (2.20) implies that

$$g(\nabla_X Y, U) = g(\nabla^*_X Y, U) + C(X, Y)g(\xi, U) = g(\nabla^*_X Y, U) = 0, \tag{4.14}$$

for any $X, Y \in \Gamma(P_{\lambda_1})$ and $U \in \Gamma(P_{\lambda_2})$. Clearly, (4.14) shows that $\nabla^*_X Y \in \Gamma(P_{\lambda_1})$, and hence P_{λ_1} is a parallel distribution. Similar reasoning can be applied to ascertain the parallelness of P_{λ_2}. As these distributions are also integrable (see Lemma 4.1), we can apply the decomposition theorem originally used by de Rham [2] to show that each leaf M' of $S(TM)$ is a product manifold $M_{\lambda_1} \times M_{\lambda_2}$. Here, M_{λ_1} and M_{λ_2} are integral manifolds of P_{λ_1} and P_{λ_2}, respectively. It is easy to show that they are totally geodesic submanifolds of M' and totally umbilic submanifolds of $\overline{M}$. In view of (2.26) and (2.27), we have

$$\begin{aligned} g(R^*(X,Y)PZ,PW) &= c(g(Y,PZ)g(X,PW) - g(X,PZ)g(Y,PW)) \\ &+B(Y,PZ)C(X,PW) + C(Y,PZ)B(X,PW) \\ &-C(X,PZ)B(Y,PW) - B(X,PZ)C(Y,PW), \end{aligned} \tag{4.15}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Letting $X = W = E_i$ and $Y = Z = E_j$ in (4.15) and using (3.4) together with the fact $2ab + 1 = 0$, we derive

$$g(R^*(E_i, E_j)E_j, E_i) = c + \frac{1}{b^2}\lambda_i\lambda_j - \frac{\alpha}{b}(\lambda_i + \lambda_j), \tag{4.16}$$

for all $1 \leq i < j \leq n$. From (4.16) we see that the sectional curvatures ς of any leaf M' of $S(TM)$ are given by

$$\varsigma(E_i, E_j) = c + \frac{1}{b^2}\lambda_i\lambda_j - \frac{\alpha}{b}(\lambda_i + \lambda_j). \tag{4.17}$$

It then follows from (4.17), (2.13) and the fact M_{λ_1} is totally geodesic in $S(TM)$ that the sectional curvature c_1 of M_{λ_1} is

$$c_1 = c + \frac{1}{b^2}\lambda_1^2 - \frac{2\alpha}{b}\lambda_1 = \left(\alpha - \frac{\lambda_1}{b}\right)^2 - \rho. \tag{4.18}$$

By a similar reasoning, the sectional curvature of M_{λ_2} is

$$c_2 = \left(\alpha - \frac{\lambda_2}{b}\right)^2 - \rho. \tag{4.19}$$

It can easily be verified that the two sectional curvatures c_1 and c_2 are constant along $S(TM)$. In fact, as $\tau = 0$, we have from (3.3) that $Xb = \alpha b\theta(X)$, for any $X \in \Gamma(TM)$. Setting $X = PX$ in this relation gives $PXb = 0$, since M is ascreen. It follows that b is constant along $S(TM)$. On the other hand, from (2.4) we have $X\alpha = d\alpha(X) = \rho\theta(X)$, for all $X \in \Gamma(T\overline{M})$. It also follows that $PX\alpha = 0$. Since by (2.13), $\rho = (\alpha^2 - c)$, we get that ρ is constant along $S(TM)$. Therefore, c_1 is constant along $S(TM)$. Similarly, c_2 is constant along $S(TM)$. Therefore, the integral manifolds M_{λ_1} and M_{λ_2} are of constant curvatures c_1 and c_2, respectively. By Theorem 3.2, M is locally a product manifold

$$\mathcal{C}_\xi \times M_{\lambda_1}(c_1) \times M_{\lambda_2}(c_2), \tag{4.20}$$

where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$, and λ_1, λ_2 satisfy the relations $(k - 1)\lambda_1 + (n-k-1)\lambda_2 = 2b\alpha(n-1)$ and $\lambda_1\lambda_2 = 2b^2(\gamma - nc)$, which follows directly from (4.4), (4.5) and the fact that $2ab + 1 = 0$. Applying E_i, where $1 \leq i \leq k$, to the first of these relations and using Lemma 4.1, we get $(n - k - 1)E_i(\lambda_2) = 0$. Similarly, $(k - 1)E_j(\lambda_2) = 0$, for $k + 1 \leq j \leq n$. Thus, both λ_1 and λ_2 are constant along $S(TM)$. Hence, by the language of [13], M is an isoparametric null hypersurface.

Next, suppose that $\gamma = c = 0$, then the second relation of (4.9) suggests that; either $\lambda_1 = 0$ or $\lambda_2 = 0$. If the first case holds, then the first relation of (4.9) leads to $\lambda_2 = \frac{2b\alpha(n-1)}{(n-k-1)}$. Consequently, (4.18) and (4.19) gives $c_1 = (\alpha^2 - \rho) = c = 0$ and $c_2 = \frac{4\alpha^2 k(n-1)}{(n-k-1)^2} > 0$. It follows as in [10] that there exist a coordinate chart $\mathcal{O}$ for each leaf M' of $S(TM)$ such that $\mathcal{O} \longmapsto \mathbb{R}^k \times \mathbb{S}^{n-k}(r)$, where $r = \frac{n-k-1}{2\alpha\sqrt{k(n-1)}}$. For the second case, we get $c_1 = \frac{4\alpha^2(n-1)(n-k)}{(k-1)^2} > 0$ and $c_2 = 0$. Thus, we have $\mathcal{O} \longmapsto \mathbb{S}^k(r) \times \mathbb{R}^{n-k}$, where $r = \frac{k-1}{2\alpha\sqrt{(n-1)(n-k)}}$. Which, together with (4.20), proves (4.10) and (4.11). Now, let $\alpha = 1$ and $\gamma = 0$. Then it follows from (2.13) and (2.5) that $\rho = 0$ and $c = 1$. Considering (4.8) with $\rho = 0$, we have $c_i = \left(1 - \frac{\lambda_i}{b}\right)^2$. Also, (4.9) with $\alpha = 1$, $c = 1$ and $\gamma = 0$ implies (4.13). Note that $c_i > 0$. In fact, if $c_i = 0$, then we have $\lambda_i = b$. Feeding this in (4.13) gives $k = n - \frac{n+1}{2n+1}$, which is contradiction as k is an integer for any given integer $n \geq 3$. Therefore, as in [10] there exist a unique mapping $\mathcal{O} \longmapsto \mathbb{S}^k(r_1) \times \mathbb{S}^{n-k}(r_2)$, where $r_1 = \frac{b}{b-\sigma}$ and $r_2 = \frac{\sigma}{\sigma - 2nb}$, respectively. This proves (4.12), which completes the proof. □

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